ITERATING MAHLO'S OPERATION

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Abstract: The principle of "collecting the universe" justifies axioms asserting the existence of large cardinals which can be "built up" by iterating Mahlo's operation. In a previous paper "schemes" were used to define iterations of length up to κ^+. This paper gives a method for defining iterations of length up to κ^{++}. Assuming GCH this is an unsurpassable limit, and the question of what ranks are achievable is complicated. Results are given which suggest that weakly compact cardinals can be built up, and give an idea of their rank. It is also argued that the notion of "built up" cardinals suggests, along with other evidence, that all sets are constructible.
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1. Introduction

In axiomatic set theory the intended universe of discourse of all sets cannot be a set (disregarding antifoundational theories which attempt to deal with the situation by less classical methods). The history of set theory has taught logicians proper precautions, and they sometimes address the issue. The notion of the totality of all sets is reasonably well behaved, and this suggests that inaccessible cardinals exist. That is, the fact that the totality of sets can be considered indicates that disallowing a V_κ satisfying second order replacement is unjustified.

In addition, higher types can be added to set theory. Continuing the considerations from above, it is apparent that these systems prove only true facts about sets. In Section 3 a system will be given where there is no formal distinction between inaccessible cardinals and higher types.

Altogether, the evidence for the existence of inaccessible cardinals seems quite strong. The principle applied might be called "collecting the universe"; any sufficiently well prescribed universe of sets can itself be collected into a set. The notion of a proper class is as before a logical device whose "ontology" may not be clear, but for which adequate formal rules are clear.

Collecting the universe provides a method for "building up" large cardinals by iterating the operation. For example, for an ordinal α the notion of α-inaccessibility may be defined by transfinite recursion. A cardinal κ is 0-inaccessible iff it is inaccessible. It is $\alpha+1$-inaccessible iff it is α-inaccessible and there are κ α-inaccessibles below it. For limit α, κ is α-inaccessible iff it is β inaccessible for every $\beta < \alpha$. It is a routine exercise to convince oneself that the existence of α-inaccessible cardinals for any ordinal α is justified by collecting the universe (further discussion is given below). Applying it once again, there are cardinals κ which are κ-inaccessible.

As pointed out in Dowd [5], the results of Gaifman [10] may be seen as justifying the existence of Mahlo cardinals by taking a sort of "limit" of iterations such as considered above. Immediately, even larger small large cardinals can be justified.

This paper presents a theory of such iterations which reaches a limit beyond which it cannot be pushed. The results suggest that weakly compact cardinals are within this limit. Section 2 contains a discussion of preliminary facts. Section 3 discusses a formal system for higher types. In Section 4 "schemes" are defined and a quick review given of Gaifman's theory as revised by Dowd [5]. In Section 5 the relation of the rank function of Jech [13] to schemes is reviewed. Section 6 gives a method for using schemes to obtain larger cardinals. In Section 7 "superschemes" are defined; these extend the iteration to an unsurpassable limit.

Section 8 discusses a restricted variety of effective superschemes. Sections 9 to 12 introduce material needed for a discussion of effective superschemes. These are defined in Section 13, and their relation to weakly compact cardinals discussed.

The justification of large cardinals by "building them up" can be seen as evidence for constructibility of all sets, in that it is inconceivable that cardinals which contradict constructibility can be built up. Section 14 considers the issue of constructibility, reviewing other evidence for it. Section 15 concludes the paper with a discussion of further research.

The notion of a "sufficiently well prescribed" collection has evolved, and presumably will continue to do so. The author believes that various cardinals claimed to have been "built up" in this paper result from sufficiently well prescribed collections, and hence their existence should be taken as an axiom of set theory.

2. Basic facts

For a set S $P(S)$ denotes the powerset of S. For an ordinal α $\mathrm{Cf}(\alpha)$ denotes the cofinality of α. Let Ord denote the ordinals and Lim the limit ordinals. Let Inac denote the (strongly) inaccessible cardinals; throughout the remainder of the paper κ denotes an inaccessible cardinal. Since it occurs so frequently, Inac_κ is used to denote $\mathrm{Inac}\cap\kappa$.

For an ordinal α, let $\cap_{\xi<\beta}$ for $\beta<\alpha$ be the operation on $P(\alpha)$ taking a sequence x_ξ, $\xi<\beta$, to $\cap_{\xi<\beta}x_\xi$. Let $\triangle$ ("diagonal intersection") be the operation taking a sequence x_ξ, $\xi<\alpha$, to $\{\gamma:\gamma\in x_\xi \text{ for } \xi<\gamma\}$. Often α is an inaccessible cardinal κ, but the more general definition is sometimes required.

Call a filter in $P(\kappa)$ normal if it is closed under the operations $\cap_{\xi<\beta}$ for $\beta<\kappa$ and $\triangle$. It is well known that this is so if it contains each "final segment" $\{\gamma:\gamma\geq\alpha\}$ where $\alpha<\kappa$, and is closed under $\triangle$. By the filter generated by a collection S is meant as usual the smallest filter containing S. An ideal I is called normal if the dual filter $\{X : X^c\in I\}$ is normal; an ideal is normal iff it is closed under unions of length less than κ, and "diagonal union" $\{\gamma:\gamma\in x_\xi \text{ for some } \xi<\gamma\}$ (the symbol $\triangledown$ is used for this).

For $X\subseteq\kappa$ let $\mathrm{Lim}(X)$ denote the set of α such that $X\cap\alpha$ is unbounded below α (in particular α must be a limit ordinal). The operation Lim on $P(\kappa)$ is monotone ($X\subseteq Y\Rightarrow\mathrm{Lim}(X)\subseteq\mathrm{Lim}(Y)$), and local ($\mathrm{Lim}(X\cap\alpha)=\mathrm{Lim}(X)\cap\alpha$). Also, $\mathrm{Lim}(\mathrm{Lim}(X))\subseteq\mathrm{Lim}(X)$ and $\mathrm{Lim}(X\cup Y)=\mathrm{Lim}(X)\cup\mathrm{Lim}(Y)$. Let $\mathrm{Cl}(X)$ denote $X\cup\mathrm{Lim}(X)$. This is readily verified to satisfy the axioms for a topological closure operator, namely, $\mathrm{Cl}(\emptyset)=\emptyset$, $X\subseteq\mathrm{Cl}(X)$, $\mathrm{Cl}(\mathrm{Cl}(X))=\mathrm{Cl}(X)$, and $\mathrm{Cl}(X\cup Y)=\mathrm{Cl}(X)\cup\mathrm{Cl}(Y)$. A set X is closed in this topology iff $\mathrm{Lim}(X)\subseteq X$.

Let $\mathrm{L}(X)$ denote $X\cap\mathrm{Lim}(X)$. Given a set of ordinals S, there is a unique increasing function on an ordinal whose range is S. Although no use will be made of the fact, Proposition 4 of Gaifman [10] (attributed to Mahlo) states that if X is a set of regular cardinals then $\mathrm{L}(X)$ is the fixed points of the increasing enumeration of X.

The abbreviation "club" is used for "closed and unbounded". It is well known that the subsets of κ which contain a club set comprise a normal filter, which is closed under L (see Jech [12]). Subsets of κ which are in the dual ideal to the club filter are called thin; that is, a set is thin if its complement contains a club set. A set which is not thin is called stationary. Thus, a subset of κ is stationary iff it intersects every club subset of κ.

For $X\subseteq Y\subseteq\kappa$, X is closed in Y (i.e., in the relative topology on Y) if $\mathrm{Lim}(X)\cap Y\subseteq X$. The typical example for this paper is that the 1-inaccessibles are closed in the inaccessibles. Say that X is club in Y if also X is unbounded. Even more generally, X can be defined to be closed in Y for any $X,Y\subseteq\kappa$, if $\mathrm{Lim}(X)\cap Y\subseteq X$; and club in Y if also $X\cap Y$ is unbounded. If Y is stationary and X is club in Y then X is stationary (given a club set C, $Y\cap\mathrm{Lim}(X)\cap C\neq\emptyset$).

For $X,Y\subseteq\mathrm{Inac}_\kappa$ define $X\subseteq_{\mathrm{t}}Y$ if $X-Y$ is thin. This relation is reflexive and transitive, and the operations $\cap_{\xi<\beta}$ and $\triangle$ respect $\subseteq_{\mathrm{t}}$ (this follows noting that $\cap X_\xi-\cap Y_\xi\subseteq\cup(X_\xi-Y_\xi)$ and $\triangle X_\xi-\triangle Y_\xi\subseteq\triangledown(X_\xi-Y_\xi)$). Also, $\triangle_\xi X_\xi\subseteq_{\mathrm{t}}X_\xi$, and if $X\subseteq_{\mathrm{t}}X_\xi$ for all $\xi<\kappa$ then $X\subseteq_{\mathrm{t}}\triangle_\xi X_\xi$. Finally, L respects $\subseteq_{\mathrm{t}}$, and the club filter respects the equivalence relation "$X\subseteq_{\mathrm{t}}Y\wedge Y\subseteq_{\mathrm{t}}X$", denoted $X\equiv_{\mathrm{t}}Y$.

For $X\subseteq\kappa$ let $\mathrm{H}(X)$ denote the set of inaccessible cardinals λ such that $\lambda\in X$ and $X\cap\lambda$ is stationary in λ. This operation (or some variation of it) is Mahlo's operation. The requirement that λ be inaccessible is often omitted, but is convenient for this paper. H satisfies $\mathrm{H}(X\cup Y)=\mathrm{H}(X)\cup\mathrm{H}(Y)$. From this it follows that H respects $\subseteq_{\mathrm{t}}$. Either L or H can be iterated; indeed in Gaifman [10] operations weaker than L are iterated. However, there are sometimes advantages to using H, for example it accomplishes a greater reduction at each step.

Theorem 1. *$X-L(X)$ is thin.*

Proof. If X is bounded the claim is clear. If $\mathrm{Lim}(X)$ is bounded then X is bounded, else an ascending chain of length ω can be chosen. If $\mathrm{Lim}(X)$ is unbounded it is club, and is disjoint from $X-\mathrm{L}(X)$.

Theorem 2. *If X is a stationary set of inaccessibles then $X-H(X)$ is stationary.*

Proof. Let C be a club set, and choose $\lambda_0\in X\cap\mathrm{Lim}(C)$. Inductively, if $\lambda_i\in\mathrm{H}(X)$ choose $\lambda_{i+1}\in X\cap\mathrm{Lim}(C)\cap\lambda_i$; the process must terminate after finitely many steps.

An ordinal α is said to be Π^1_1-indescribable if for all Π^1_1 formulas $\Phi(P)$ and $P\subseteq V_\alpha$, $\models_{V_\alpha}\Phi(P)$ implies $\models_{V_\beta}\Phi(P)$ for some $\beta<\alpha$. The notation is abused as follows; P denotes a second order variable or a class interpreting it, and in a transitive submodel it denotes the intersection with the model. It is well-known (and follows from facts given below) that α must be an inaccessible cardinal, and β may be required to be one.

These cardinals will also be called "weakly compact", although this is often taken to mean a closely related class. If κ is Π^1_1-indescribable, $\Phi(P)$ is Π^1_1, $P \subseteq V_\kappa$, and $S \subseteq \kappa$ say that $\Phi(P)$ enforces S if $\models_{V_\kappa} \Phi(P)$ and $\{\alpha < \kappa : \models_{V_\alpha} \Phi(P)\} \subseteq S$; and say that S is Π^1_1-enforceable if it is enforced by some $\Phi(P)$.

For κ Π^1_1-indescribable, let E be the collection of Π^1_1-enforceable $S \subseteq \kappa$. Then E is a proper normal filter which contains Inac_κ and is closed under H. This is well known; for convenience a proof is sketched. First, $\kappa \in \mathrm{Inac}$ iff κ is Π^1_0-indescribable; we omit a proof of this and refer the reader to Drake [7] (Theorem 9.1.3). Second, there is a Π^1_1 formula $\mathrm{Tru}(c, X)$ such that for any Π^1_1 formula $\Phi(X)$ and any $\alpha \in \mathrm{Lim}$ with $\alpha > \omega$,

$$\models_{V_\alpha} \forall X(\Phi(X) \Leftrightarrow \mathrm{Tru}(\ulcorner\Phi\urcorner, X))$$

where $\ulcorner\Phi\urcorner$ is the Gödel number of Φ (which is an integer); again the reader is referred to Drake [7] (Theorem 9.1.9) for details. Third, if $X \in E$ then X is stationary; indeed, if X is enforced by $\Phi(P)$ and $C \subseteq \kappa$ is club then "C is unbounded and $\Phi(P)$" holds in V_κ. But then it holds in V_α for some $\alpha < \kappa$, whence $\alpha \in C \cap X$.

The formula enforcing Inac_κ is "$\exists x(x = \omega)$ and $\forall x(P(x)$ is a set) and $\forall F \forall x$(if F is a function then $F[x]$ is a set)". If X is enforced by $\Phi(P)$ then $\mathrm{H}(X)$ is enforced by "$\Phi(P)$ and $\forall Y(Y$ club implies $Y \cap X$ nonempty)". Suppose $X_\xi \subseteq \kappa$ for $\xi < \kappa$ is is enforced by $\Phi_\xi(P_\xi)$. Let $P = \{\langle\xi, p\rangle : p \in P_\xi\}$; let $Q = \{\langle\xi, \ulcorner\Phi_\xi\urcorner\rangle\}$; and let $\mathrm{Tru}'(\xi, c, X)$ be true iff $\mathrm{Tru}(c', X)$ is true, where if $c = \ulcorner\Psi\urcorner$ then $c' = \ulcorner\Psi'\urcorner$ where Ψ' is Ψ with all subformulas $w \in X$ replaced by $\langle\xi, w\rangle \in X$. Then $\triangle_{\xi<\kappa} X_\xi$ is enforced by

$$\exists x(x = \omega) \wedge \forall x \exists y(x \in y) \wedge \forall\xi\forall c(\langle\xi, c\rangle \in Q \Rightarrow \mathrm{Tru}'(\xi, c, P)).$$

For the case $\cap_{\xi<\mu} X_\xi$, replace $\exists x(x = \omega)$ by $\exists x(x = \mu)$, and $\forall\xi$ by $\forall\xi < \mu$.

In various circumstances κ may be replaced by Ord, provided the necessary definitions are given; sometimes these are immediate. For example, $\cap_{\xi<\beta}$ for any ordinal β, L, and H are clearly defined on classes of ordinals. The "locality" property shows that the definition on some κ is just the restriction of the general definition. For another example, a class C of ordinals is said to be Π^1_1-enforceable if it contains the Π^1_1-indescribable cardinals, and for every Π^1_1-indescribable cardinal κ, $C \cap \kappa$ is Π^1_1-enforceable in V_κ.

3. Higher types.

This section is peripheral to the main topics of this paper, but provides some relevant background. The development of ZFC using proper classes as an extension of the language (see Takeuti and Zaring [22]) gives many laws satisfied by proper classes in a restricted context. The system called Bernays-Gödel (or von Neumann-Bernays-Gödel) gives further laws, and its "impredicative" version further still (see Tharp [21] for both systems). It is easy to generalize these methods to consider objects of type α for any ordinal α; such a system, called ZFCT, is given in Dowd [5]. Various other systems for higher type objects have been given, for example in Freidman [8].

Adding higher types to set theory yields new perspectives on foundational issues, provides new methods in set theory and other areas, and introduces new research in its own right. Kreisel [14] asks the question, "what are the proper laws (the 'logic') satisfied by the intensional element of the crude mixture". The answer to this question is provided by adding higher types to first order set theory. The sets comprise the ground domain. The proper classes are the objects of type 1, etc. As the above examples indicate, the basic axioms which these higher type objects should satisfy are clear and indeed have been given in various contexts.

There seems little doubt that these systems prove only true facts about sets. The question of their semantics is however filled with difficulties. This is especially true if one accepts the principle of collecting the universe. Any time one considers the universe of discourse as a collection, one can consider it as a set. On the other hand, not only does it appear to make sense to consider the totality of sets, it appears to make sense to erect a type structure on top of it. Further, any distinction in the methods used in collecting the universe into a set, and into a totality, could only be made by giving a formal system.

Here a system for higher types is considered, which further illustrates these issues. It considers types higher than those of ZFCT; there are various reasons for wishing to consider such types. A simple system accomplishing this adds to ZFC a constant $\mathbf{v}$ for the universe. The axioms for $\mathbf{v}$ are

1. $x \in \mathbf{v} \wedge y \in x \Rightarrow y \in \mathbf{v}$;
2. $x \in \mathbf{v} \Rightarrow P(x) \in \mathbf{v}$; and
3. $\mathbf{f} : \mathbf{v} \mapsto \mathbf{v} \wedge x \in \mathbf{v} \Rightarrow \mathbf{f}[x] \in \mathbf{v}$ where $\mathbf{f} \subseteq \mathbf{v}$ is a function and $\mathbf{f}[x]$ is the setwise image.

ZFV is used to denote this system. It is clearly equivalent to ZFC+"there exists an inaccessible cardinal". Indeed the existence of an inaccessible cardinal is equivalent to $\exists x U(x)$ where $U(x)$ is the conjunction of the axioms for $\mathbf{v}$. By first order logic $\vdash_{\mathrm{ZFV}} F(\mathbf{v})$ iff $\vdash_{\mathrm{ZFC}} U(x) \Rightarrow F(x)$, and $\vdash_{\mathrm{ZFV}} F$ iff $\exists x U(x) \vdash_{\mathrm{ZFV}} F$.

ZFV considers ZFC as applying to objects, which are logically more abstruse than sets; they are some kind of "ideal" or "meta" sets, which are added on top of the universe for convenience. That this is acceptable, at least from the point of view of proving theorems about sets, is a manifestation of the legitimacy of the principle of collecting the universe. Further, just as there is no end to the cumulative hierarchy, there is no end to the hierarchy of abstruseness of the metasets.

In ZFV, theorems about sets are those relativized to $\mathbf{v}$. Axioms concerning the existence of large cardinals can be relativized to $\mathbf{v}$ and given in their second order form (Levy's reflection axiom is a good example). On the other hand they can be given in their first order form for the abstruse universe. The relations between such axioms is clearly of interest, but will not be considered here. Note that the axiom of replacement of ZFV states that the type structure has a certain closure property, which seems to be an assumption within reason. Indeed, the failure of the types of ZFCT to have certain closure properties is one reason for wishing to extend the type hierarchy.

ZFV is a very convenient system also, in that special arguments for higher types are no longer necessary. Arguments in higher types and arguments in the first few levels of the cumulative hierarchy above an inaccessible cardinal now can be seen as identical rather than merely resembling each other. By a metaset of type $\alpha > 0$ is meant one of rank $\mathbf{o} + \alpha$ where $\mathbf{o}$ is the rank of $\mathbf{v}$. Note, however, that we might wish to observe that methods less powerful than ZFV are sufficient. A good example is provided by higher type categories, which are considered in Dowd [6]; indeed, the system ZFCT_ω is adequate for proving the properties of these that might be of interest to a category theorist. Another point worth mentioning is that the machinery of ZFCT can be added to ZFV.

The question of constructible metasets is readily dispensed with in ZFV. The predicate L is defined as usual, and interpreted as applying to metasets. A constructible proper class is a constructible metaset of rank $\mathbf{o}$. Tharp [21] gives a more involved definition, which can be carried out entirely within $V_{\mathbf{o}+1}$. We sketch a proof that the definitions are equivalent.

Define an A-structure for the language of set theory to be a structure $\langle M, E\rangle$ where M is a proper class, which satisfies foundation (i.e., any subset of M contains an E-minimal element), extensionality, and the axioms stating closure under the Gödel operations. An A-structure $\langle M, E\rangle$ is called a B-structure iff it satisfies a sentence σ which states that $V = L$; see Tharp [21].

Say that a B-structure $\langle M, E\rangle$ and a $z \in M$ determine a class Z, as follows. For $x \in \mathbf{l}$ (the constructible sets of $\mathbf{v}$), $x \in Z$ iff there is a $y \in M$ with yEz, and the transitive closure of x (in the universe) is isomorphic to the transitive closure of y (in $\langle M, E\rangle$). Say that a proper class is T-constructible iff it is determined in this way.

Theorem 3. *A proper class is T-constructible iff it is constructible.*

Proof. (Sketch). Let α denote a limit ordinal with $\alpha < \mathbf{o}+$. First, it can be shown that a B-structure is isomorphic to some L_α, and every L_α is isomorphic to a B-structure. Second, a proper class is constructible iff it is in some L_α. Third, replacing the B-structure by the L_α, the condition on x, y implies that they are equal. Clearly every constructible class is T-constructible. On the other hand if Z is T-constructible then $Z = z \cap \mathbf{l}$ and Z is constructible.

As another example of the utility of ZFV, a class of indescribable cardinals is described which is stronger than the T-indescribables of Dowd [5]. V_κ for κ inaccessible may be considered as a structure for the language of ZFV, by interpreting $\mathbf{v}$ as V_κ and the objects of the type hierarchy as sets of rank $\geq \kappa$. Some limit must be placed on the types, so that the overall structure is a set.

For example the limit might be considered to be $\mathbf{o}$, yielding the T-indescribables of Dowd [5]. The formulas must be relativized to objects of rank $< \mathbf{o} + \mathbf{o}$. If the limit is considered to be the next largest inaccessible the formulas may be unrestricted. These cardinals might be called I-indescribable. These and similarly defined cardinals seem worthy of investigation, but this is omitted here, except for a more complete statement of the definition.

Suppose $\Phi(P)$ is a sentence in the language of ZFV with free class variables added; $\kappa \in \mathrm{Inac}$; and $\mu \in \mathrm{Inac}$ the the next largest inaccessible cardinal after κ. Then $\Phi(P)$ is true in V_κ if it is true in V_μ as a formula of ZFC, with $\mathbf{v}$ interpreted as V_κ, and P a subset of V_κ (the latter requirement being most consistent with standard types of indescribability). Finally κ is I-indescribable if whenever $\Phi(P)$ is true in V_κ, it is true in V_λ for some $\lambda \in \mathrm{Inac}_\kappa$.

4. Schemes

For an ordinal α call an increasing unbounded sequence in α a *limiting* sequence. A *prescheme* in κ is defined to be a pair $\langle \sigma, \chi\rangle$ where $\sigma < \kappa^+$ is a successor ordinal (which may be called the length of Σ), and χ is a function with domain $\sigma \cap \mathrm{Lim}$, such that $\chi(\alpha)$ is a limiting sequence in α.

A prescheme provides information for an iteration; the limiting sequences indicate the inputs at limit stages. Preschemes are in fact a special type of well-founded tree, the latter arising from infinitary expressions. Indeed, consider a descending chain $\alpha_0 \ni \alpha_1 \ni \cdots \ni \alpha_n$ of ordinals in κ as the sequence $\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle$. Define an ordinal tree in κ to be a set T of such chains, which is closed under prefix and where each chain starts at the same ordinal.

Let $*$ denote concatenation of sequences, or of an ordinal and a sequence, etc. The following terminology may be introduced.

- The chains of T may be called nodes.
- The label of a node is its last ordinal.
- Node $n * \alpha$ is a son of node n for an ordinal α.
- A subtree of T is those sequences m where for some sequence n and ordinal α, $n * m$ are the chains of T starting with $n * \alpha$.
- An ordinal α occurs in T if it occurs in some chain.

Say that an ordinal tree is *regular* if it satisfies the following restrictions (the first is redundant, but stated explicitly anyway). For n a node labeled α,

1. if $\alpha = 0$ n has no sons;
2. if $\alpha = \beta + 1$ n has a unique son, and its label is β;
3. if $\alpha \in \mathrm{Lim}$ the labels of the sons of n in increasing order form a limiting sequence; and
4. subtrees with the same root label are identical.

It is easily seen that for regular trees, if α occurs in T and $\beta < \alpha$ then β occurs in T. The label of the root is the ordinal usually associated with a well-founded tree (called the length in Moschovakis [17]).

Theorem 4. *There are maps $\Sigma \mapsto T$ from preschemes to regular ordinal trees, and $T \mapsto \Sigma$ from regular ordinal trees to preschemes, which are inverse to each other.*

Proof. $\Sigma \mapsto T$ may be defined recursively; for example if $\alpha \in \mathrm{Lim}$ the labels of the sons of the node labeled α in T are the elements of the limiting sequence for α in Σ. $T \mapsto \Sigma$ may also be defined recursively; for example if $\alpha \in \mathrm{Lim}$ the sequence of Σ "up to" α is the union of the sequences up to the labels of the sons of the node labeled α, with α appended.

Restrictions may be placed on the limiting sequences; note that the preceding equivalence holds when preschemes and regular trees have the same restriction imposed. Call a prescheme a scheme if the domain of $\chi(\alpha)$ equals some $\delta < \kappa$ (resp. κ) when $\mathrm{Cf}(\alpha) < \kappa$ (resp. $\mathrm{Cf}(\alpha) = \kappa$).

The term "cofinal" is often used instead of "limiting"; in this paper this will be reserved for a limiting sequence whose domain is $\mathrm{Cf}(\alpha)$. Placing this restriction on the limiting sequences has important consequences, as will be seen. The more general definition is considered, however, in particular because it has lower logical complexity.

A scheme $\Sigma = \langle \sigma, \chi \rangle$ in κ can be used to iterate an operation $f : P(\kappa) \mapsto P(\kappa)$. For $\alpha < \sigma$ an operation $f^\alpha : P(\kappa) \mapsto P(\kappa)$ is defined recursively by its action on $X \subseteq \kappa$ as follows. For convenience let α_ξ denote $\chi(\alpha)(\xi)$.

1. $f^0(X) = X$;
2. $f^{\alpha+1}(X) = f(f^\alpha(X))$;
3. $f^\alpha(X) = \cap_{\xi<\delta} f^{\alpha_\xi}(X)$ if $\alpha \in \mathrm{Lim}$, $\mathrm{Cf}(\alpha) < \kappa$, and δ is the domain of $\chi(\alpha)$;
4. $f^\alpha(X) = \triangle_{\xi<\kappa} f^{\alpha_\xi}(X)$ if $\alpha \in \mathrm{Lim}$ and $\mathrm{Cf}(\alpha) = \kappa$.

Let σ^- denote α where $\sigma = \alpha + 1$. Let f_Σ denote f^{σ^-}. For some other useful notation, let $\mathcal{S}^\kappa$ denote the set of schemes in κ, and $\mathcal{S}^\kappa_\sigma$ those whose length is σ.

It is readily verified that for f monotone, there is a proper normal filter containing X and closed under f iff $f_\Sigma(X) \neq \emptyset$ for all $\Sigma \in \mathcal{S}^\kappa$.

Theorem 5. *Suppose $Y \subseteq \kappa$ is stationary; then $\mathrm{L}_\Sigma(Y)$ is club in Y for all $\Sigma \in \mathcal{S}^\kappa$.*

Proof. The proof is by induction on Σ. If X is club in Y then clearly $\mathrm{L}(X)$ is closed in Y. Also, $\mathrm{Lim}(X)$ is club, so $Y \cap \mathrm{Lim}(X)$ is unbounded. Suppose X_ξ is club in Y for $\xi < \delta$ where $\delta < \kappa$. Clearly $\cap_{\xi<\delta} X_\xi$ is closed in Y. Also, $\cap_{\xi<\delta}\mathrm{Lim}(X_\xi)$ is club, so $Y \cap (\cap_{\xi<\delta}\mathrm{Lim}(X_\xi))$ is unbounded. Suppose X_ξ is closed in Y for $\xi < \kappa$, and suppose $\alpha \in \mathrm{Lim}(\triangle_\xi X_\xi) \cap Y$. Let α_η be a sequence in $\triangle_\xi X_\xi$ converging to α. If $\xi < \alpha$ then some suffix of the sequence converges in X_ξ to α, so $\alpha \in X_\xi$. But this shows that $\alpha \in \triangle_\xi X_\xi$. The argument for unboundedness is similar to the intersection case.

Theorem 6. *Suppose $Y \subseteq \kappa$ is not stationary; then $L_\Sigma(Y) = \emptyset$ for some $\Sigma \in \mathcal{S}^\kappa$.*

Proof. Let $Z \subseteq \kappa$ be a club set disjoint from Y. Enumerate Z in natural order as $\langle \alpha_\gamma : \gamma < \kappa \rangle$. Choose any scheme in $\mathcal{S}^\kappa_{\kappa+1}$ where the limiting sequence for κ is $\langle \alpha_\gamma \rangle$. By induction $Y_\alpha \cap \alpha = \emptyset$ for $\alpha < \kappa$. Thus for $\alpha \in \mathrm{Lim}\ \alpha \notin Y_\kappa$, and thus $\mathrm{L}(Y_\kappa) = \emptyset$.

An inaccessible cardinal κ is said to be Mahlo if the inaccessible cardinals below κ comprise a stationary set. By the preceding, this is so iff $\mathrm{L}_\Sigma(\mathrm{Inac}_\kappa) \neq \emptyset$ for all $\Sigma \in \mathcal{S}^\kappa$, iff $\mathrm{L}_\Sigma(\mathrm{Inac}_\kappa)$ is stationary for all $\Sigma \in \mathcal{S}^\kappa$. As promised in the introduction, this can be seen as justifying the existence of Mahlo cardinals via collecting the universe. First, one argues by collecting the universe that $\mathrm{L}_\Sigma(\mathrm{Inac})$ is nonempty for any scheme Σ in Ord. Second, by collecting the universe there is an inaccessible cardinal κ such that $\mathrm{L}_\Sigma(\mathrm{Inac}_\kappa) \neq \emptyset$ for all $\Sigma \in \mathcal{S}^\kappa$. Finally, such a κ is a Mahlo cardinal.

The definition of schemes in Ord will not be stated, since it is is given in Dowd [5] (the role of σ is played by a well-order on Ord, and a scheme can be "coded" as a class). That $\mathrm{L}_\Sigma(\mathrm{Inac})$ is nonempty for any scheme Σ is a more precise version of the notion that one can always "keep going" in defining orders of inaccessibility. Collecting the universe is a reflection principle, that if the universe has a property of a character such that the principle applies, then there is a cardinal κ such that V_κ has the property. Possibly formal characters can be given adequate to build up various cardinals, but initially intuition can be relied on.

The first application of the principle concludes that there is an inaccessible cardinal, since Ord has the properties of one. It can next be concluded that the inaccessible cardinals do not form a sequence in Ord whose domain is an ordinal, since such a universe is not well-closed with respect to further collection. That is, Ord has the properties of a 1-inaccessible, and so there is a 1-inaccessible cardinal, and $\mathrm{L}(\mathrm{Inac}) \neq \emptyset$. But then the 1-inaccessibles are not enumerated by an ordinal, so there is a 2-inaccessible. Indeed if it has been concluded that $\mathrm{L}_\Sigma(\mathrm{Inac}) \neq \emptyset$ for Σ of length σ, then it can concluded that $\mathrm{L}_\Sigma(\mathrm{Inac}) \neq \emptyset$ for Σ of length $\sigma + 1$.

Suppose $\beta <$ Ord is an ordinal, $A_\xi < \mathrm{Ord}^+$ for $\xi < \beta$ is a class coding an ordinal, A is the limit of the ascending chain A_ξ, $\mathrm{L}_\Sigma(\mathrm{Inac}) \neq \emptyset$ for $\sigma^- = A_\xi$ for all $\xi < \beta$. Again appealing to the unlimitedness of the closure, it can be concluded that $\mathrm{L}_\Sigma(\mathrm{Inac}) \neq \emptyset$ for the resulting Σ of length $A + 1$. In the simplest case, if n-inaccessibles exist for all n then Ord is ω-inaccessible and so there are ω-inaccessible cardinals.

If β-inaccessibles exist for all $\beta <$ Ord then Ord is Ord-inaccessible, so there are cardinals λ which are λ-inaccessible, so $\mathrm{L}_\Sigma(\mathrm{Inac}) \neq \emptyset$ for $\sigma^- = \mathrm{Ord}$. Generalizing, $\mathrm{L}_\Sigma(\mathrm{Inac}) \neq \emptyset$ for all Σ.

More details of the justification would be desirable. However it seems possible that this would be facilitated by further research, for example more formal versions of the above considerations. We omit this here, claiming only that the reasonability of assuming the existence of Mahlo cardinals has been demonstrated. Indeed (as observed in Dowd [5]), it is reasonable to strengthen the principle of collecting the universe, to require that a universe have the Mahlo property that the inaccessible cardinals are stationary. Thus, in building up large cardinals H may be iterated rather than L.

Given a scheme $\Sigma \in \mathcal{S}^\kappa_\sigma$, the notation X_α will frequently be used for $\mathrm{H}^\alpha(X)$, where $\alpha < \sigma$.

Theorem 7. *If $X \subseteq_t Y$ for $X, Y \subseteq \mathit{Inac}_\kappa$ and $\Sigma \in \mathcal{S}^\kappa_\sigma$, then $X_\alpha \subseteq_t Y_\alpha$.*

Proof. This follows by induction on α, using observations made in Section 2.

Theorem 8. *Suppose $X \subseteq \mathrm{Inac}_\kappa$ and $\Sigma, \Sigma' \in \mathcal{S}^\kappa_\sigma$; then $H_\Sigma(X) \equiv_t H_{\Sigma'}(X)$.*

Proof. The proof is by induction on σ^-. Let $X_\alpha = \mathrm{H}^\alpha(X)$ for Σ and $X'_\alpha = \mathrm{H}^\alpha(X)$ for Σ'. Let α_ξ (α'_ξ) be the limiting sequence, with domain δ (δ') if $\mathrm{Cf}(\alpha) < \kappa$. The cases $\alpha = 0$ and successor α are left to the reader. If $\mathrm{Cf}(\alpha) < \kappa$, given $\eta < \delta'$ there is a $\xi < \delta$ with $\alpha_\xi \geq \alpha'_\eta$, and $\cap_{\xi<\delta} X_{\alpha_\xi} \subseteq X_{\alpha_\xi} \subseteq_t X'_{\alpha'_\eta}$; the claim follows by further such arguments. If $\mathrm{Cf}(\alpha) = \kappa$ the argument is similar, except $\triangle_{\xi<\kappa} X_{\alpha_\xi} \subseteq_t X_{\alpha_\xi} \subseteq_t X'_{\alpha_{\eta'}}$.

A similar observation to Theorem 8 is made in Baumgartner, Taylor, Wagon [3]. Note also that if $\sigma < \kappa$ then $\mathrm{H}_\Sigma(X) = \mathrm{H}_{\Sigma'}(X)$. Define the scheme rank of an inaccessible cardinal κ to be the supremum of the σ such that $\mathrm{H}_\Sigma(\mathrm{Inac}_\kappa)$ is stationary for all $\Sigma \in \mathcal{S}^\kappa_\sigma$. The rank is at least 0 if κ is inaccessible; at least 1 if κ is Mahlo; etc. By Theorem 8, $\mathrm{H}_\Sigma(\mathrm{Inac}_\kappa)$ is stationary for all $\Sigma \in \mathcal{S}^\kappa_\sigma$ iff it is stationary for some $\Sigma \in \mathcal{S}^\kappa_\sigma$.

A standard definition states that for an ordinal α, $M_0 = \mathrm{Inac}$; $M_{\alpha+1} = \mathrm{H}(M_\alpha)$ (H being the operation on classes); and for $\alpha \in \mathrm{Lim}\ M_\alpha = \cap_{\beta<\alpha} M_\beta$. It is not difficult to show that for $\alpha < \kappa$, $\kappa \in M_\alpha$ iff its rank is at least α (indeed, κ has rank $\geq \alpha + 1$ iff $\mathrm{H}^\alpha(\mathrm{Inac}_\kappa)$ is stationary iff $\kappa \in \mathrm{H}^{\alpha+1}(\mathrm{Inac})$). The terminology "hyper-Mahlo" is used for inaccessible cardinals κ which are in M_κ, etc., but the "inside" definition is clearly preferable to the "outside" definition. Note that given κ, the smallest member of M_κ is no less than κ, and equals κ iff κ is hyper-Mahlo, iff the rank of κ is at least κ.

It will turn out to be useful to have a method for relativising the effect of a scheme $\Sigma \in \mathcal{S}^\kappa_\sigma$ to an inaccessible cardinal $\lambda < \kappa$. Given $X \subseteq \kappa \cap \mathrm{Inac}$, for $\alpha < \sigma$ define subsets $X_{\alpha,\lambda} \subseteq \lambda$ by the following recursion. As usual, at a limit ordinal α α_ξ denotes the limiting sequence and δ its domain if $\mathrm{Cf}(\alpha) < \kappa$.

1. $X_{0,\lambda} = X_0 \cap \lambda$.

2. $X_{\alpha+1,\lambda} = \mathrm{H}(X_{\alpha,\lambda})$.

3. If $\mathrm{Cf}(\alpha) < \kappa$ and $\delta < \lambda$ $X_{\alpha,\lambda} = \cap_{\xi<\delta} X_{\alpha_\xi,\lambda}$.

4. If $\mathrm{Cf}(\alpha) < \kappa$ and $\lambda \le \delta < \kappa$ $X_{\alpha,\lambda} = X_0 \cap \lambda$.

5. If $\mathrm{Cf}(\alpha) = \kappa$ $X_{\alpha,\lambda} = \triangle_{\xi<\lambda} X_{\alpha_\xi,\lambda}$

Let $\mathrm{H}_{\Sigma,\lambda}(X \cap \lambda)$ denote $X_{\sigma^-,\lambda}$.

Theorem 9. *Given $\Sigma \in \mathcal{S}^\kappa_\sigma$, an inaccessible cardinal $\lambda < \kappa$, $X \subseteq \kappa \cap Inac$, and $\alpha < \sigma$, $X_\alpha \cap \lambda \subseteq_t X_{\alpha,\lambda}$. For given α equality holds except on a thin set of $\lambda \in Inac_\kappa$.*

Proof. A more general result is proved in Theorem 16; the reader might provide a proof of this by induction on α as an exercise.

Theorem 10. *Given a scheme $\Sigma \in \mathcal{S}^\kappa$ and $\lambda \in Inac_\kappa$, a scheme $\Sigma' = \langle \sigma', \chi' \rangle$ in $\mathcal{S}^\lambda$ can be defined so that $H_{\Sigma'}(X \cap \lambda) \equiv {}_t H_{\Sigma,\lambda}(X)$ for all X.*

Proof. The function $r(\Sigma, \lambda)$ such that $r(\Sigma, \lambda) + 1$ is the length of Σ' may be defined by recursion on Σ as follows; α, δ, etc., are as usual, and $r(\Sigma, \lambda)$ is denoted $r(\alpha, \lambda)$.

$$\begin{aligned}
r(0,\lambda) &= 0 \\
r(\alpha+1,\lambda) &= r(\alpha,\lambda)+1 \\
r(\alpha,\lambda) &= \sup\{r(\alpha_\xi,\lambda) : \xi < \delta\} \text{ if } \mathrm{Cf}(\alpha) < \kappa \text{ and } \delta < \lambda \\
&= 0 \text{ if } \mathrm{Cf}(\alpha) < \kappa \text{ and } \delta \ge \lambda \\
r(\alpha,\lambda) &= \sup\{r(\alpha_\xi,\lambda) : \xi < \lambda\} \text{ if } \mathrm{Cf}(\alpha) = \kappa
\end{aligned}$$

The only case which requires clarification is the last. If Y_β is a reordering of X_α then $\triangle X_\alpha = \triangle Y_\beta$; and if there are fewer than λ distinct X_α the intersection may be taken. These facts are noted in Baumgartner, Taylor, Wagon [3] and are left to the reader.

A question of interest is whether there is a $\Sigma \in \mathcal{S}^\kappa_\sigma$ such that $\rho(\Sigma, \lambda) = \sup\{\rho(\Phi, \lambda) : \Phi \in \mathcal{S}^\kappa_\sigma\}$. Indeed, a system of such can be constructed by induction on $\alpha < \kappa^+$, together with the function $r(\alpha, \lambda) = r(\Sigma, \lambda)$, by giving a limiting sequence for each $\alpha \in \mathrm{Lim}$. In fact a cofinal sequence will be given. Suppose $\mathrm{Cf}(\alpha) < \kappa$ and let $\delta = \mathrm{Cf}(\alpha)$. If $\delta \ge \lambda$ let α_ξ be an arbitrary cofinal sequence. If $\delta < \lambda$ let $\alpha' = \sup\{r(\beta, \lambda) : \beta < \alpha\}$. If $\alpha' = r(\beta, \lambda)$ for some $\beta < \alpha$ take α_ξ to be any cofinal sequence where β is an element. Otherwise let α_ξ be cofinal in $\sup \alpha_\xi$ and such that $r(\alpha_\xi)$ limits to α'; α_ξ must limit to α, else $\alpha' = r(\beta, \lambda)$ for some $\beta < \alpha$.

To summarize the preceding paragraph, given κ, λ, and α there is a scheme $\Sigma \in \mathcal{S}^\kappa_{\alpha+1}$, such that the induced scheme in $\Sigma \in \mathcal{S}^\lambda$ has maximum length among such. The function $r(\alpha, \lambda)$ giving its length satisfies the recursion:

$$\begin{aligned}
r(0,\lambda) &= 0 \\
r(\alpha+1,\lambda) &= r(\alpha,\lambda)+1 \\
r(\alpha,\lambda) &= \sup\{r(\beta,\lambda) : \beta < \alpha\} \text{ if } \mathrm{Cf}(\alpha) < \kappa \text{ and } \mathrm{Cf}(\alpha) < \lambda \\
&= 0 \text{ if } \mathrm{Cf}(\alpha) < \kappa \text{ and } \mathrm{Cf}(\alpha) \ge \lambda \\
r(\alpha,\lambda) &= \sup\{r(\beta,\lambda) : \beta < \alpha\} \text{ if } \mathrm{Cf}(\alpha) = \kappa
\end{aligned}$$

To conclude this section some minor observations will be made. Given a limiting sequence in an ordinal α, this function is continuous iff its range is closed. An increasing continuous function is often called "normal". The increasing enumeration of the closure of the range will be called the "normal closure". If Σ' is obtained from $\Sigma \in \mathcal{S}^\kappa$ by enlarging limiting sequences, in general only $\mathrm{H}_{\Sigma'}(X) \equiv {}_t\mathrm{H}_\Sigma(X)$ for all X can be concluded. If, however, Σ' is obtained by replacing each limiting sequence by its normal closure then $\mathrm{H}_{\Sigma'}(X) = \mathrm{H}_\Sigma(X)$ for all X. The proof is by induction and left to the reader.

If the definition of scheme is changed, to require that a limiting sequence for α be cofinal, then the results of this section all hold mutatis mutandis. In the proof of Theorem 10a a subsequence of α_ξ with domain $\mathrm{Cf}(\alpha)$ must be taken if necessary. For the fact of the preceding paragraph, it must be noted that the domain of the normal closure is still $\mathrm{Cf}(\alpha)$. To see this, note that any prefix of the enumeration of the closure has a domain of cardinality less than $\mathrm{Cf}(\alpha)$.

5. Jech rank

An alternative method for specifying the "Mahlo-ness" of a cardinal has been given by Jech [13]. Let $<$ be a well-founded transitive relation on a set D. D may be partitioned into "levels" by the transfinite recursion, D_ν equals the minimal elements of $D - \cup_{\mu<\nu} D_\mu$. For $x \in D$ the rank $\rho(x)$ of x is defined to be the value of ν such that $x \in D_\nu$. The following facts are readily verified.

- If $\nu < \rho(y)$ then there is a z with $z < y$ and $\rho(z) = \nu$. (If not then y is not greater than any minimal element of $D - \cup_{\xi<\nu} D_\xi$, so y is a minimal element, a contradiction.)
- If $x < y$ then $\rho(x) < \rho(y)$. (If $\rho(x) = \rho(y)$ then x and y are incomparable. If $\rho(x) > \rho(y)$ then there is a y' with $\rho(y') = \rho(y)$ and $y' < x$; but then $y' < y$, a contradiction.)
- If x, y are such that $z < y \Rightarrow z < x$ then $\rho(y) \le \rho(x)$. (This follows by transfinite induction on $\rho(y)$.)

For X, Y stationary subsets of Inac_κ define $X > Y$ if $X \subseteq_{\mathrm{t}} \mathrm{H}(Y)$. This relation is transitive.

Theorem 11. *The relation $<$ is well-founded.*

Proof. Suppose $X_0 > X_1 > X_2 > \cdots$ is an infinite descending chain. Let C_i be a club set disjoint from $X_i - \mathrm{H}(X_{i+1})$, and let $C = \cap \mathrm{Lim}(C_i)$. $\mathrm{H}(X_1)$ is stationary, so choose $\lambda \in \mathrm{H}(X_1) \cap C$. Then below λ X_1 is stationary and every C_i is club. It follows that $X_1 \cap \lambda > X_2 \cap \lambda > \cdots$ is an infinite descending chain below λ; continuing inductively leads to a contradiction.

For the remainder of this section $>$ denotes this order, and ρ the corresponding rank function. Note that if $X \subseteq_{\mathrm{t}} Y$ then $\rho(X) \ge \rho(Y)$ (since then $Z < Y$ implies $Z < X$). Note also that if Inac_κ is stationary then it is a minimal element; this follows by well-foundedness.

Theorem 12. *Suppose that $Y = H_\Sigma(Inac_\kappa)$ is stationary where $\Sigma \in \mathcal{S}^\kappa_\sigma$. Then $\rho(Y) = \sigma^-$, and if $Z \subseteq Inac_\kappa$ and $\rho(Z) \ge \rho(Y)$ then $Z \subseteq_t Y$.*

Proof. The proof is by induction on Σ, the basis $Y = \mathrm{Inac}$ being observed above. First, let S be those $Z \in \mathrm{Inac}_\kappa$ with $\rho(Z) \ge \alpha$, and suppose $Y \in S$ is such that $Z \subseteq_{\mathrm{t}} Y$ whenever $Z \in S$. Then Y is minimal among the sets of S, so $\rho(Y) = \alpha$; for if $Z \in S$ satisfied $Y > Z$ then $Z \subseteq_{\mathrm{t}} Y$, or $Y \subseteq_{\mathrm{t}} \mathrm{H}(Z)$, a contradiction. Suppose $W = \mathrm{H}_\Sigma(\mathrm{Inac}_\kappa)$ and $Y = \mathrm{H}(W)$; let $\alpha = \rho(W)$. If $Z \subseteq \mathrm{Inac}_\kappa$ and $\rho(Z) > \alpha$ then there is a $V \subseteq \mathrm{Inac}_\kappa$ such that $\rho(V) = \alpha$ and $Z > V$. By induction $V \subseteq_{\mathrm{t}} W$, so $Z \subseteq_{\mathrm{t}} \mathrm{H}(V) \subseteq_{\mathrm{t}} \mathrm{H}(W) = Y$. Also by induction $\rho(Y) > \alpha$. By the first observation this proves the claim for Y. Suppose $\alpha \in \mathrm{Lim}$ and α_ξ is a limiting sequence in α. Let W_ξ be $\mathrm{H}^{\alpha_\xi}(\mathrm{Inac}_\kappa)$. If $Z \in S$ then by induction $Z \subseteq_{\mathrm{t}} W_\xi$ for any ξ; it follows in either case of $\mathrm{Cf}(\alpha)$ that $Z \subseteq_{\mathrm{t}} Y$, and the theorem follows.

If $\rho(\mathrm{H}(X)) = \rho(X) + 1$ when $\mathrm{H}(X)$ is stationary were true, this could be used in the proof of Theorem 12; however this is false in L. Let M_α be as in Section 4. If $\kappa \in M_3$ and there is a stationary set $S \subseteq (M_0 - M_1) \cap \kappa$ such that $\mathrm{H}(S)$ is thin, then $X = (M_1 \cap \kappa) \cup S$ is a counterexample. The existence of such an S follows in L for κ not weakly compact by Theorem VII.1.2′ of Devlin [4].

Writing $\rho_S(\kappa)$ for the scheme rank, and $\rho_J(\kappa)$ for $\sup\{\rho(X) + 1 : X \subseteq \mathrm{Inac}_\kappa, X \text{ stationary}\}$, we have $\rho_S(\kappa) \ge \sigma$ iff $\mathrm{H}_\Sigma(\mathrm{Inac}_\kappa)$ is stationary iff $\rho_J(\kappa) \ge \sigma$. Thus, the two ranks are the same below rank κ^+. The scheme rank is at most κ^+, and $\rho_J(\kappa) \ge \kappa^+$ iff $\rho_S(\kappa) \ge \kappa^+$ iff $\mathrm{H}_\Sigma(\mathrm{Inac}_\kappa)$ is stationary for all schemes $\Sigma \in \mathcal{S}^\kappa$. Such cardinals are called greatly Mahlo. Let H_* be the map from $P(\kappa)$ to $P(\kappa)$ where $\lambda \in \mathrm{H}_*(X)$ iff $\lambda \in \mathrm{Inac} \cap X$ and $\mathrm{H}_\Sigma(X \cap \lambda)$ is stationary for all $\Sigma \in \mathcal{S}^\lambda$. This operation is clearly local. The greatly Mahlo cardinals below κ are those in $\mathrm{H}_*(\mathrm{Inac}_\kappa)$.

Theorem 13. *Suppose $X \subseteq Inac_\kappa$ where $\kappa \in Inac$.*

a. *$\lambda \in H_*(X)$ iff $H_\Sigma(X \cap \lambda) \ne \emptyset$ for all $\Sigma \in \mathcal{S}^\lambda$.*

b. *$\lambda \in H_*(X)$ iff $H_{\Sigma,\lambda}(X)$ is stationary below λ for all $\Sigma \in \mathcal{S}^\kappa$.*

c. *If $\lambda \in H_*(X)$ then $H_\Sigma(X) - H_*(X)$ is stationary below λ for all for all $\Sigma \in \mathcal{S}^\lambda$.*

d. *For any $\Sigma \in \mathcal{S}^\kappa$, $H_*(X) \subseteq_t H_\Sigma(X)$.*

e. *If $X \subseteq_t Y$ then $H_*(X) \subseteq_t H_*(Y)$.*

Proof. Part a is left to the reader. Part b follows by Theorem 10. For part c, let C be club. Since $\mathrm{H}_\Sigma(X)$ is stationary there is a $\mu < \lambda$ in $\mathrm{Lim}(C) \cap \mathrm{H}_\Sigma(X)$. If $\mu \notin \mathrm{H}_*(X)$ we are done; otherwise we may continue inductively. Part d is proved by induction on $\alpha = \sigma^-$, the basis $\alpha = 0$ being immediate. Suppose $\lambda \in \mathrm{H}_*(X)$ and inductively $\lambda \in \mathrm{H}_\Sigma(X)$ except on a thin set. By part b $\mathrm{H}_{\Sigma,\lambda}(X \cap \lambda)$ is stationary below λ, so by Theorem 9 $\mathrm{H}_\Sigma(X) \cap \lambda$ is stationary below λ, except for a thin set. Thus, $\lambda \in \mathrm{H}\mathrm{H}_\Sigma(X)$, except for a thin set. If $\lambda \in \mathrm{H}^{\alpha_\xi}(X)$ except for a thin set for $\xi < \delta < \kappa$, then $\lambda \in \cap_{\xi<\beta} \mathrm{H}^{\alpha_\xi}(X)$ except for a thin set; and similarly for diagonal intersection. For part e, if $\lambda \in \mathrm{H}_*(X) - \mathrm{H}_*(Y)$ then for some $\Sigma \in \mathcal{S}^\lambda$ $\mathrm{H}_\Sigma(X \cap \lambda)$ is stationary and $\mathrm{H}_\Sigma(Y \cap \lambda)$ is thin. By Theorem 7, $X - Y$ is stationary in λ. If this is so for a stationary set of λ then $X - Y$ is stationary.

Theorem 14. *Suppose $S \subseteq_t H_\Sigma(X)$ for all $\Sigma \in \mathcal{S}^\kappa$; then $S \subseteq_t H_*(X)$.*

Proof. Suppose there is a stationary set $S_2 \subseteq S$ such that for $\lambda \in S_2$ there is a $\Sigma_\lambda \in \mathcal{S}^\lambda$ with $\mathrm{H}_{\Sigma_\lambda}(X \cap \lambda)$ thin. We claim that there is a $\Sigma \in \mathcal{S}^\kappa$ such that $\mathrm{H}_{\Sigma,\lambda}(X \cap \lambda) \subseteq \mathrm{H}_{\Sigma_\lambda}(X \cap \lambda)$ for all $\lambda \in S_2$. Then $\mathrm{H}_{\Sigma,\lambda}(X \cap \lambda)$ is thin for $\lambda \in S_2$, and so $\mathrm{H}_\Sigma(X) \cap \lambda$ is thin for $\lambda \in S_3$, where $S_3 \subseteq S_2$ is stationary, and so $\lambda \notin \mathrm{H}\mathrm{H}_\Sigma(X)$ for $\lambda \in S_3$. The general idea for constructing Σ is to concatenate the schemes for Σ_λ given by Theorem 10b; and add a cofinal sequence visiting the ends of the concatenated schemes. However this does not work at those λ where there are λ members of S below it. For these, the segment for λ is extended to a diagonal intersection which visits the required point; the final cofinal sequence then includes this cofinal sequence rather than the end point, and omits the end point.

6. Schemes and functionals.

For an ordinal α a hierarchy of functionals can be defined by recursion as follows.

1. $\mathcal{H}^0_\alpha = P(\alpha)$.
2. $\mathcal{H}^{\theta+1}_\alpha$ is the functions $f : \mathcal{H}^\theta_\alpha \mapsto \mathcal{H}^\theta_\alpha$.
3. For a limit ordinal θ $\mathcal{H}^\theta_\alpha$ is those functions $f : \cup_{\eta<\theta}\mathcal{H}^\eta_\alpha \mapsto \cup_{\eta<\theta}\mathcal{H}^\eta_\alpha$ which are "graded", meaning that for $\eta < \theta$ $f[\mathcal{H}^\eta_\alpha] \subseteq \mathcal{H}^\eta_\alpha$.

$\mathcal{H}^\theta_\alpha$ for $\theta > 0$ is closed under composition. The operations $\cap$ and $\triangle$ may be defined on $\mathcal{H}^\theta_\alpha$ "recursively pointwise"; that is, $(\cap f_\xi)(x) = \cap f_\xi(x)$, and similarly for $\triangle$.

Let $\mathcal{L}^0_\alpha = \mathcal{H}^0_\alpha$, and for $x \in \mathcal{L}^0_\alpha$ and $\beta < \alpha$ let $x \restriction \beta = x \cap \beta$. Inductively say that $f : \mathcal{L}^\theta_\alpha \mapsto \mathcal{L}^\theta_\alpha$ is local if $f(x \restriction \beta) = (f(x)) \restriction \beta$ for all x in $\mathcal{L}^\theta_\alpha$ and all $\beta < \alpha$, and let $\mathcal{L}^{\theta+1}_\alpha$ denote the collection of these. For $f \in \mathcal{L}^{\theta+1}_\alpha$ let $f \restriction \beta$ be defined by $(f \restriction \beta)(x) = (f(x)) \restriction \beta$. The definition of a local functional and its restrictions when θ is a limit ordinal is similar and left to the reader.

The following facts are readily verified.

- If $\theta > 0$ then $\mathcal{L}^\theta_\alpha$ is closed under composition.
- $\mathcal{L}^\theta_\alpha$ is closed under $\cap$ and $\triangle$.
- The map $f \mapsto f \restriction \beta$ induces a map from $\mathcal{L}^\theta_\alpha$ to $\mathcal{L}^\theta_\beta$. This map commutes with $\cap$ and $\triangle$, and composition if $\theta > 0$.
- $f = \cup_{\beta<\alpha} f \restriction \beta$.

Given a particular $\kappa \in \text{Inac}$, write $\mathcal{H}^\theta$ for $\mathcal{H}^\theta_\kappa$ and similarly for $\mathcal{L}^\theta$. Although defined more generally, $\theta < \kappa$ is assumed (as noted below, this ensures that each functional f can be coded as a class in V_κ, and for $\alpha < \kappa$ $f \restriction \alpha$ is a set).

Call a descending chain $\theta_1 > \cdots > \theta_t$ of ordinals a predecessor chain if $\theta_{i-1} = \theta_i - 1$ when θ_i is a successor ordinal, and $\theta_t = 0$. Writing f^θ for a member of $\mathcal{L}^\theta$ for $\theta < \kappa$, f^θ_* is defined by the requirement that whenever $\theta > \theta_1 > \cdots$ is a predecessor chain, λ is in $f^\theta_*(f^{\theta_1})\cdots(f^1)(x)$ iff $\lambda \in \text{Inac}$, $\lambda > \theta$, and $f^\theta_\Sigma(f^{\theta_1})\cdots(f^1)(x) \neq \emptyset$ for all schemes $\Sigma \in \mathcal{S}^\lambda$.

Let I^θ be the map $f^{\theta'} \mapsto f^{\theta'}_*$ where $\theta' = \theta - 1$ for successor θ, and any $\theta' < \theta$ for limit θ. For $\theta < \kappa$ define κ to be $\mathcal{H}^\theta$-Mahlo iff $I^\theta_\Sigma(I^{\theta_1})\cdots(I^1)(\text{H})(\text{Inac}_\kappa) \neq \emptyset$ for all schemes $\Sigma \in \mathcal{S}^\kappa$ and all predecessor chains $\theta > \theta_1 > \cdots$. The case $\theta = 0$ is included; the subscript Σ is on H, and these cardinals are the greatly Mahlo cardinals. The $\mathcal{H}^1$-Mahlo cardinals are defined in Gaifman [10]; they are also considered in Gloede[11] and Dowd [5]. Gaifman [10] mentions that one can continue further; the importance of doing so was indicated in Dowd [5].

Results later in the paper place these cardinals in a larger context. They illustrate the limitations of schemes, in that increasing the rank requires progressively more complicated artifices. To conclude this section an outline will be given of a proof that the $\mathcal{H}^\theta$-Mahlo cardinals for $\theta < \kappa$ are Π^1_1-indescribable will be given. Again, in later sections more general methods are given.

In proving the claim, it is necessary to demonstrate that certain statements are Π^1_1; this is facilitated by observing that certain relations are Π^1_0. Now, a local functional f can be coded as the class of pairs $\langle \alpha, f \restriction \alpha \rangle$. Writing $h = f(g)$ as a Π^1_0 formula is completely straightforward, involving a quantification on α of some set-theoretic formulas involving the class free variables. The case $f \in \mathcal{L}^1$ is slightly special, since plain classes can be coded as themselves.

Again referring the reader to Dowd [5] for schemes in Ord, such a scheme can be coded as a class by first coding the ordinal as a well order on V; the limiting sequences can then be coded, as a class since there are at most κ of them. There is a Π^0_1 formula in a single second order variable defining the codes of schemes. A witness W that $f_\Sigma = g$, for f, g codes of elements of $\mathcal{L}^\theta$ and Σ a code for a scheme, is an appropriate sequence of elements of $\mathcal{L}^\theta$, and can be coded as a class of triples $\langle \xi, \alpha, f_\xi \restriction \alpha \rangle$. Further, the formula "$W$ witnesses $f_\Sigma = g$" is Π^1_0.

Theorem 15. *Suppose κ is Π^1_1-indescribable, and $\theta < \kappa$. Then the set of $\mathcal{H}^\theta$-Mahlo cardinals below κ is Π^1_1-enforceable.*

Proof. The sentence Φ attesting to enforceability is "for all Σ, for all predecessor chains $\theta > \theta_1 > \cdots$, $I^\theta_\Sigma(I^{\theta_1})\cdots(I^1)(\mathrm{H})(\mathrm{Inac}) \neq \emptyset$" (the conjunct "$\exists x(x = \theta)$" should be added). This is Π^1_1 because the statement that $g = (f^\theta \mapsto f^\theta_*) \restriction \alpha$ for $\alpha, \theta < \kappa$ is first order in V_κ. It remains to show that Φ is true in V_κ. Let $\mathcal{E}^0 = \mathcal{E}$, let $\mathcal{E}^{\theta+1}$ be the elements of $L^{\theta+1}$ under which $\mathcal{E}^\theta$ is closed, and for $\theta \in \mathrm{Lim}$ let $\mathcal{E}^\theta$ be the elements of L^θ under which $\mathcal{E}^{\theta'}$ is closed for all $\theta' < \theta$. It follows straightforwardly by induction that $\mathcal{E}^\theta$ is closed under composition, $\cap$, and $\triangle$. To complete the proof it must be shown that $I^\theta(I^{\theta_1})\cdots(I^1)(\mathrm{H})(\mathrm{Inac}) \neq \emptyset$ is true in V_κ, or that $I^{\theta_1}_*\cdots(I^1)(\mathrm{H})(\mathrm{Inac}) \neq \emptyset$ is. This follows by the definition and induction.

7. Superchemes

A *superscheme* in κ is defined to be a pair $\langle\sigma, \chi\rangle$ where $\sigma < \kappa^{++}$ is a successor ordinal, and χ is a function with domain $\sigma \cap \mathrm{Lim}$, such that $\chi(\alpha)$ is a limiting sequence in α; further the domain of $\chi(\alpha)$ must equal some $\delta < \kappa$ (resp. κ, κ^+) when $\mathrm{Cf}(\alpha) < \kappa$ (resp. $\mathrm{Cf}(\alpha) = \kappa$, $\mathrm{Cf}(\alpha) = \kappa^+$). Let $\mathfrak{S}^\kappa$ denote the set of superschemes in κ, and $\mathfrak{S}^\kappa_\sigma$ those whose length is σ. Let $\mathfrak{C}^\kappa$ denote the set of superschemes in κ, all of whose limiting sequences are cofinal; and $\mathfrak{C}^\kappa_\sigma$ those whose length is σ.

A superscheme can be used to iterate Mahlo's operation. Given $\Sigma \in \mathfrak{S}^\kappa_\sigma$ and $X \subseteq \kappa \cap \mathrm{Inac}$, for $\alpha < \sigma$ and $\lambda \in \kappa \cap \mathrm{Inac}$ subsets $X_\alpha \subseteq \kappa$ and $X_{\alpha,\lambda} \subseteq \lambda$ are defined, as for schemes. Indeed, the recursion is the same for cases other than $\mathrm{Cf}(\alpha) = \kappa^+$, as follows.

	X_α	$X_{\alpha,\lambda}$
$\alpha = 0$	X	$X \cap \lambda$
$\alpha + 1$	$\mathrm{H}(X_\alpha)$	$\mathrm{H}(X_{\alpha,\lambda})$
$\mathrm{Cf}(\alpha) < \kappa$	$\cap_{\xi<\delta} X_{\alpha_\xi}$	$\cap_{\xi<\delta} X_{\alpha_\xi,\lambda}$ if $\delta < \lambda$
		$X \cap \lambda$ if $\lambda \leq \delta < \kappa$
$\mathrm{Cf}(\alpha) = \kappa$	$\triangle_{\xi<\kappa} X_{\alpha_\xi}$	$\triangle_{\xi<\lambda} X_{\alpha_\xi,\lambda}$

There are various possibilities for the additional case $\mathrm{Cf}(\alpha) = \kappa^+$. For example, $\lambda \in X_\alpha$ iff $\lambda \in X \cap \kappa$; and

1. $X_{\alpha_\xi,\lambda}$ is stationary for all $\xi < \lambda^+$,
2. $X_{\alpha_\xi,\lambda}$ is stationary for all $\xi < \kappa^+$, or
3. $X_{\gamma,\lambda}$ is stationary for all $\gamma < \alpha$;

and $X_{\alpha,\lambda} = X_\alpha \cap \lambda$. The notation $X^{(i)}_\alpha$ is used to distinguish between the possibilities. Observe that $X^{(3)}_\alpha \subseteq X^{(2)}_\alpha \subseteq X^{(1)}_\alpha$; in case 2 $\nu \in X_{\alpha,\lambda}$ iff $\nu \in \mathrm{Inac}_\lambda$ and $X_{\alpha_\xi,\nu}$ is stationary for all $\xi < \nu^+$; and in case 3 the limiting sequences at stages of cofinality κ^+ are not used. As before $\mathrm{H}_\Sigma(X)$ denotes X_{σ^-}.

At stages of cofinality κ^+, superschemes generalize the way in which the greatly Mahlo cardinals are obtained. One wishes to ensure that λ has "sufficiently high" rank with respect to superschemes in λ. The definitions approximate, in different ways, this requirement using "outside" superschemes, i.e., superschemes in κ. Until it becomes clear which definition (if any) is preferable, all will be considered.

As an initial observation, $X_\alpha \subseteq_{\mathrm{t}} X$ and $X_{\alpha,\lambda} \subseteq_{\mathrm{t}} X \cap \lambda$ for all $\lambda \in \mathrm{Inac}_\kappa$; this holds in all three cases. The proof is a simple induction.

Theorem 16. *Suppose $\Sigma \in \mathfrak{S}^\kappa_\sigma$, $X, Y \subseteq \mathit{Inac}_\kappa$, and $X \subseteq_t Y$. Then $X_\alpha \subseteq_t Y_\alpha$; and for any $\lambda \in \mathit{Inac}_\kappa$ such that $X \cap \lambda \subseteq_t Y \cap \lambda$, $X_{\alpha,\lambda} \subseteq_t Y_{\alpha,\lambda}$. This holds for all three cases of the definition.*

Proof. The proof is by induction on σ. The basis $\alpha = 0$ is immediate. All cases of the induction but $\mathrm{Cf}(\alpha) = \kappa^+$ follow by earlier observations. If $\lambda \in X_\alpha - Y_\alpha$ then (in case 1 of the definition) there is a $\xi < \lambda^+$ such that $X_{\alpha_\xi,\lambda}$ is stationary and $Y_{\alpha_\xi,\lambda}$ is thin. It follows that $X - Y$ is stationary below λ. If this occurs for a stationary set of λ then $X - Y$ is stationary. The second claim follows similarly. The argument for the other cases of the definition is similar.

Theorem 17. *Suppose $\Sigma \in \mathfrak{S}^\kappa$ and $\nu < \lambda < \kappa$ where $\nu, \lambda \in \mathit{Inac}_\kappa$.*

a. *$X_\alpha \cap \lambda \subseteq_t X_{\alpha,\lambda}$, and for fixed α equality holds except for a thin set of $\lambda < \kappa$.*

b. *$X_{\alpha,\lambda} \cap \nu \subseteq_t X_{\alpha,\nu}$, and for fixed α equality holds except for a thin set of $\nu < \lambda$.*

This holds for all three cases of the definition.

Proof. First part a is proved by induction on α. As usual, at a limit ordinal α α_ξ denotes the limiting sequence and δ its domain. If $\alpha = 0$, $X_0 \cap \lambda = X_{0,\lambda}$ by definition. For successor ordinals, $X_{\alpha+1} \cap \lambda = \mathrm{H}(X_\alpha) \cap \lambda = \mathrm{H}(X_\alpha \cap \lambda) \subseteq_{\mathrm{t}} \mathrm{H}(X_{\alpha,\lambda}) = X_{\alpha+1,\lambda}$; and if $X_\alpha \cap \lambda = X_{\alpha,\lambda}$ then $X_{\alpha+1} \cap \lambda = X_{\alpha+1,\lambda}$. Suppose $\mathrm{Cf}(\alpha) < \kappa$. If $\delta < \lambda$ $X_\alpha \cap \lambda = (\cap_{\xi<\delta} X_{\alpha_\xi}) \cap \lambda = \cap_{\xi<\delta}(X_{\alpha_\xi} \cap \lambda) \subseteq_{\mathrm{t}} \cap_{\xi<\delta} X_{\alpha_\xi,\lambda} = X_{\alpha,\lambda}$; if $\lambda \leq \delta$ the next to last expression is replaced by $X_0 \cap \lambda$; and if $\lambda > \delta$ and $X_{\alpha_\xi} \cap \lambda = X_{\alpha_\xi,\lambda}$ for all $\xi < \delta$ (which holds except for a thin set) then equality holds. Suppose $\mathrm{Cf}(\alpha)=\kappa$. $X_\alpha\cap\lambda=(\triangle_{\xi<\kappa}X_{\alpha_\xi})\cap\lambda=\triangle_{\xi<\lambda}(X_{\alpha_\xi}\cap\lambda)\subseteq_{\mathrm{t}}\triangle_{\xi<\lambda}X_{\alpha_\xi,\lambda}=X_{\alpha,\lambda}$; and if $X_{\alpha_\xi}\cap\lambda=X_{\alpha_\xi,\lambda}$ for all $\xi<\lambda$ (which holds except for a thin set) then equality holds. Suppose $\mathrm{Cf}(\alpha)=\kappa^+$. $X_{\alpha,\lambda}=X_\alpha\cap\lambda$ by definition. Part b is proved by a similar induction, left to the reader. When $\mathrm{Cf}(\alpha)<\kappa$ there are 3 cases, $\delta<\nu$, $\nu\le\delta<\lambda$, and $\lambda\le\delta$.

Theorem 18. *Suppose $\Sigma\in\mathfrak{S}^\kappa$ and $\beta\le\alpha<\sigma$. Then $X_\alpha\subseteq_t X_\beta$. This holds for case 3 of the definition.*

Proof. Note that the case $\beta=\alpha$ is immediate so $\beta<\alpha$ may be assumed. The proof is by induction on α, the basis $\alpha=0$ being immediate. As usual, at a limit ordinal α α_ξ denotes the limiting sequence and δ its domain. For successor ordinals, if $\beta\le\alpha$ then $X_{\alpha+1}\subseteq X_\alpha\subseteq_{\mathrm{t}}X_\beta$. Suppose $\mathrm{Cf}(\alpha)<\kappa$. If $\beta<\alpha$ then $\beta<\alpha_\xi$ for some ξ and $X_\alpha\subseteq X_{\alpha_\xi}\subseteq_{\mathrm{t}}X_\beta$. Suppose $\mathrm{Cf}(\alpha)=\kappa$. If $\beta<\alpha$ then $\beta<\alpha_\xi$ for some ξ and $X_\alpha\subseteq_{\mathrm{t}}X_{\alpha_\xi}\subseteq_{\mathrm{t}}X_\beta$. Suppose $\mathrm{Cf}(\alpha)=\kappa^+$. The claim is proved by induction on β, the basis $\beta=0$ being immediate. For $\beta+1$, suppose $\lambda\in X_\alpha$. Then except for a thin set of λ, $\lambda\in X_\beta$. Also, $X_{\beta,\lambda}$ is stationary, so except for a thin set of λ $X_\beta\cap\lambda$ is stationary. Thus, except for a thin set of λ, if $\lambda\in X_\alpha$ then $\lambda\in HX_\beta=X_{\beta+1}$. The cases of intersection and diagonal intersection are as in the proof of Theorem 13.d. Finally, for $\mathrm{Cf}(\beta)=\kappa^+$, if $\lambda\in X_\alpha$ then $X_{\gamma,\lambda}$ is stationary for $\gamma<\alpha$, and a fortiori for $\gamma<\beta$, so $\lambda\in X_\beta$; that is, $X_\alpha\subseteq X_\beta$.

Theorem 19. *Suppose $X\subseteq Inac_\kappa$, $\Sigma\in\mathfrak{C}^\kappa_\sigma$, and $\Sigma'\in\mathfrak{S}^\kappa_\sigma$. Then $H_\Sigma(X)\subseteq_t H_{\Sigma'}(X)$. This holds for all three cases of the definition.*

Proof. Let $X_{\alpha,\lambda}$ ($X'_{\alpha,\lambda}$) be the sets for Σ (Σ'), with other notation as in the proof of Theorem 8. By induction on α, $X_\alpha\subseteq_{\mathrm{t}}X'_\alpha$ and $X_{\alpha,\lambda}\subseteq_{\mathrm{t}}X'_{\alpha,\lambda}$ for all λ. The cases other than $\mathrm{Cf}(\alpha)=\kappa^+$ for the first claim are as in the proof of Theorem 8. These cases for the second claim are similar. When $\mathrm{Cf}(\alpha)<\kappa$ observe that $\delta\le\delta'$; if $\delta'<\lambda$ the argument is as before, and the other cases are trivial. Finally, in the case $\mathrm{Cf}(\alpha)=\kappa^+$, from the second claim if $\lambda\in X_\alpha$ then $\lambda\in X'_\alpha$; and similarly for $\nu\in X_{\alpha,\lambda}$.

From the foregoing case 3 of the definition seems best behaved, although various questions remain open; for the remainder of the section case 3 is assumed. Define the superscheme rank of an inaccessible cardinal κ to be the supremum of the σ such that $\mathrm{H}_\Sigma(\mathrm{Inac}_\kappa)$ is stationary for all $\Sigma\in\mathfrak{S}^\kappa_\sigma$. It is easily seen that if the scheme rank of κ is less than κ^+ then the superscheme rank equals it; let $\rho(\kappa)$ denote the superscheme rank. Theorem 19 has the consequence that $\rho(\kappa)\ge\sigma$ iff $\mathrm{H}_\Sigma(\mathrm{Inac}_\kappa)$ is stationary for some $\Sigma\in\mathfrak{C}^\kappa_\sigma$.

Theorem 20. *If GCH holds then every inaccessible cardinal has superscheme rank less than κ^{++}.*

Proof. By defining a cofinal sequence for each $\alpha\in\mathrm{Lim}\cap\kappa^{++}$, a chain of superschemes of length κ^{++} can be defined. If GCH holds, $\mathrm{H}_\Sigma(\mathrm{Inac}_\kappa)$ cannot be distinct for every Σ in the chain, from which it follows that $\mathrm{H}_\Sigma(\mathrm{Inac}_\kappa)$ must be thin for some Σ in the chain.

It is clearly of great interest what ranks can be achieved, without making any assumptions that cannot be justified by collecting the universe. A better understanding of superschemes should be the subject of further research; for this paper we have contented ourselves with a partial understanding. Questions left to further research include how Theorems 10 and 14 generalizes to superschemes; whether the Jech rank equals the superscheme rank; and whether the superscheme rank is the same if case 2 of the definition is used. This section concludes with some remarks on the last question.

Let P_i be the defining property of $\lambda\in X^{(i)}_\alpha$ for $i=2,3$ for α with $\mathrm{Cf}(\alpha)=\kappa^+$; that is,

P_2: $\forall\xi<\kappa^+, X^{(2)}_{\alpha_\xi,\lambda}$ is stationary

P_3: $\forall\gamma<\alpha, X^{(3)}_{\gamma,\lambda}$ is stationary

By induction on α, P_3 implies P_2, $X^{(3)}_\alpha\subseteq X^{(2)}_\alpha$, and $X^{(3)}_{\alpha,\lambda}\subseteq X^{(2)}_{\alpha,\lambda}$.

Let A be the statement that P_2 implies P_3. If A holds for all λ then by induction on α, $X^{(3)}_\alpha=X^{(2)}_\alpha$, and $X^{(3)}_{\alpha,\lambda}=X^{(2)}_{\alpha,\lambda}$ for all λ.

If A holds for almost all λ then by induction on α, $X^{(3)}_\alpha\equiv_{\mathrm{t}}X^{(2)}_\alpha$. Also, if $X^{(3)}_\alpha\equiv_{\mathrm{t}}X^{(2)}_\alpha$, then $X^{(3)}_{\alpha,\lambda}\equiv_{\mathrm{t}}X^{(2)}_{\alpha,\lambda}$ for almost all λ. If not, there is a stationary set of λ such that for a stationary set of $\nu<\lambda$, $X^{(2)}_{\alpha,\nu}-X^{(3)}_{\alpha,\nu}$ is stationary; from this, $X^{(2)}_\alpha-X^{(3)}_\alpha$ is stationary, a contradiction.

A can fail; for example suppose κ is the smallest greatly Mahlo cardinal, $\Sigma\in\mathfrak{S}^\kappa_{\kappa^++1}$, and $\alpha=\kappa^+$. Choose $\lambda<\kappa$ and let α_ξ run through values of cofinality at least λ. In this case, $X^{(3)}_\alpha=\emptyset$; we leave it open whether $X^{(2)}_\alpha$ must be thin, or more generally if for any superscheme, A holds for almost all (i.e., all but a thin set of) λ. If the latter is not true, it could still be the case that, given α with $\mathrm{Cf}(\alpha)=\kappa^+$, there is a $\Sigma\in\mathfrak{S}^\kappa_{\alpha+1}$ such that A holds for almost all $\lambda<\kappa$. It could even be true that there is a $\Sigma\in\mathfrak{S}^\kappa_{\alpha+1}$ such that A holds for all λ; but this seems less likely.

Let B be the statement that for all $\gamma<\alpha$ there is a $\xi<\kappa^+$ such that $X^{(2)}_{\alpha_\xi,\lambda}\subseteq_{\mathrm{t}}X^{(3)}_{\gamma,\lambda}$. Note that B implies A. The same questions as in the preceding paragraph can be asked for B.

8. P-superschemes.

A sequence of progressively faster growing ordinal functions may be defined by the following recursion, where $n \geq 3$.

$$\begin{aligned} f_1(\alpha, \beta) &= \alpha + \beta \\ f_2(\alpha, \beta) &= \alpha \cdot \beta \\ f_n(\alpha, 0) &= 1 \\ f_n(\alpha, \beta + 1) &= f_{n-1}(f_n(\alpha, \beta), \alpha) \\ f_n(\alpha, \beta) &= \sup\{f_n(\alpha, \beta') : \beta' < \beta\} \text{ if } \beta \in \mathrm{Lim} \end{aligned}$$

Faster growing functions can be defined by letting n be an arbitrary ordinal, and faster still using multiple recursion such as in Veblen [23], but this is omitted here.

Lemma 21.

a. *$f_n(\alpha, \beta) \geq \alpha$ if $\alpha \geq 2$ and $\beta \geq 2$.*

b. *$f_n(\beta + 1, \alpha) > f_n(\beta, \alpha)$ for all β; and $\alpha \geq 0$ if $n = 1$, $\alpha \geq 1$ if $n = 2$, and $\alpha \geq 2$ if $n \geq 3$.*

Proof. The claims may be proved by induction on n, in the order given. Details are left to the reader.

Thus, as a function of β $f_n(\alpha, \beta)$ is normal (increasing and continuous), with the provision of part b. Let κ^+_P denote $\sup\{f_n(\kappa^+, \beta) : \beta < \kappa^+, n < \omega\}$.

If $\beta < \kappa^{+2}$ there are unique $\beta_1, \beta_2 < \kappa^+$ such that $\beta = \kappa^+ \cdot \beta_2 + \beta_1$. If $\beta < \kappa^+_P$ and $\beta \geq \kappa^{+2}$ there is a unique $n \geq 3$ and $\beta_n < \kappa^+$ such that $f_n(\kappa^+, \beta_n) < \beta < f_n(\kappa^+, \beta_n + 1)$; also, $\beta_n \geq 2$. Let $\alpha_{n-1} = f_n(\kappa^+, \beta_n)$; then $f_{n-1}(\alpha_{n-1}, 1) \leq \beta < f_{n-1}(\alpha_{n-1}, \kappa^+)$. There is a unique $\beta_{n-1} < \kappa^+$ such that $f_{n-1}(\alpha_{n-1}, \beta_{n-1}) \leq \beta < f_{n-1}(\alpha_{n-1}, \beta_{n-1} + 1)$; also $\beta_{n-1} \geq 1$. Continuing, values $\beta_n, \beta_{n-1}, \ldots, \beta_1$ may be obtained, such that

$$\beta = f_1(\cdots f_{n-1}(f_n(\kappa^+, \beta_n), \beta_{n-1}) \cdots \beta_1)$$

where $\beta_i < \kappa^+$, $\beta_n \geq 2$, $\beta_i \geq 1$ for $2 \leq i < n$, and $\beta_1 \geq 0$. Let $I(\beta)$ be 1 if $\beta_1 > 0$, otherwise the smallest i such that $\beta_i > 1$.

For $\beta \in \kappa^+ \cap \mathrm{Lim}$ let β_ξ be a fixed cofinal sequence for β. Suppose $\beta \in \kappa^+_P$. If $I(\beta) = n$ (this includes the case $I(\beta) = 1$ when $\beta < \kappa^{+2}$), if $\beta_n \in \mathrm{Lim}$ let $\beta_\xi = f_n(\kappa^+, \beta_{n,\xi})$; and if $\beta_n = \gamma + 1$ let $\beta_\xi = f_{n-1}(f_n(\kappa^+, \gamma), \xi)$ for $\xi < \kappa^+$. In the remaining case, $1 \leq I(\beta) < n$, let $\beta_\xi = f_i(\alpha_i, \beta_{i,\xi})$ where $i = I(\beta)$.

Thus, a limiting sequence has been defined for each $\beta \in \kappa^+_P \cap \mathrm{Lim}$, and hence a superscheme of length σ has been defined for each $\sigma < \kappa^+_P$. Such a superscheme will be called a P-superscheme. If there is an increasing unbounded function from β to α then $\mathrm{Cf}(\beta) = \mathrm{Cf}(\alpha)$. Using this it is not difficult to show that the limiting sequences just defined are cofinal.

Conjecture 22. *Given $\kappa \in$ Inac and $\theta < \kappa$, κ is $\mathcal{H}^\theta$-Mahlo iff $\rho(\kappa) \geq f_4(\kappa^+, 1 + \theta)$.*

The proof of this might be straightforward if lengthy, but is omitted here because further basic facts about superschemes might facilitate it. The idea is as follows. For $\Sigma \in \mathfrak{S}^\kappa_{\alpha+1}$, f^Σ may be roughly denoted as f^α. Then $f^\alpha \circ f^\beta = f^{\beta+\alpha}$, $(f^\alpha)^\beta = f^{\alpha \cdot \beta}$, and $f_2^\alpha(f_1) = (f_2(f_1))^\alpha$. Indeed, operations on superschemes can be defined such that is rigorously true. $(f \mapsto f_*)(f) = f_*$ is f^{κ^+}. Each term of the form $I^\theta_\Sigma(I^{\theta_1}) \cdots (I^1)(\mathrm{H})$ corresponds to a superscheme.

9. Review of admissible ordinals.

Another specialized class of superschemes will be defined in Section 13. In aid of this some recursion theoretic machinery will be developed in Sections 9 to 11, enabling a coding of ordinals greater than κ^+ in $P(\kappa)$ to be given in Section 12. To begin with, basic properties of admissible ordinals will be reviewed. These were discovered in the 1960's to be the proper setting for recursion theory in ordinals other than ω. In this paper only the case of a cardinal is of interest, but the basic theory is the same in the general case. In this section some facts are stated without proof; proofs can be found in Devlin [4] or Barwise [1].

If S is a structure in some language, and Φ is a formula in the language of S with parameters from S and free variables $x_1, \ldots, x_n$, Φ defines the relation which holds at elements $x_1, \ldots, x_n$ of S iff Φ does. Let $\mathrm{Def}(S)$ denote the definable relations, with $\mathrm{Def}_1(S)$ denoting the subsets. A set S may be considered as a structure for the language of set theory in this context. A Δ_0 formula is one whose quantifiers are of the form $\exists u \in v$ or $\forall u \in v$, which may be reduced to a formula of the language of set theory using a well known abbreviation.

Let L_α be the sets defined sets by the following recursion.

1. $L_0 = \emptyset$.

2. $L_{\alpha+1} = \mathrm{Def}_1(L_\alpha)$.

3. For $\alpha \in \mathrm{Lim}$, $L_\alpha = \cup_{\beta<\alpha} L_\beta$.

The class L of constructible sets equals $\cup_{\alpha \in \mathrm{Ord}} L_\alpha$. L_α is a transitive set, with $L_\alpha \cap \mathrm{Ord} = \alpha$. If $\beta < \alpha$ then $L_\beta \in L_\alpha$. If $x \in L_\alpha$ and $y \in x$ then $y \in L_\beta$ for some $\beta < \alpha$. If $\alpha \in \mathrm{Lim}$ then L_α is amenable, where a set is amenable if it is transitive; closed under pairing, union, and Cartesian product; and satisfies Δ_0-separation. For $\alpha \geq \omega$ $|L_\alpha| = |\alpha|$.

A limit ordinal α is said to be admissible if L_α satisfies Δ_0-collection

- $\forall x \exists y\, \Phi \Rightarrow \forall u \exists v \forall x \in u \exists y \in v\, \Phi$

where Φ is a Δ_0 formula. Φ may contain free variables other than x and y, which are implicitly universally quantified.

Admissible ordinals are stages of the constructibility hierarchy at which L_α has various closure properties that make it a suitable setting for recursion theory. In particular, if f is a function defined by a Σ_1 formula with parameters, and $x \in L_\alpha$, then $f[x]$ (indeed $f \restriction x$) is in L_α. Also, the Σ_1 definable predicates are closed under bounded universal quantification.

An ordinal α is called a δ-number if it is closed under ordinal multiplication, and an ϵ-number if it is closed under ordinal exponentiation; see Monk [16] for basic properties of these. An admissible ordinal is an ϵ-number. A proof of this illustrates some common issues regarding Σ_1 definable functions. There is a system of axioms KP in the language of set theory, such that L_α satisfies KP if (and only if) α is admissible. Suppose the function F on V is defined by a Σ_1 formula $\Phi(x, y)$, where $\forall x \exists y \Phi$ is provable in KP. Then for admissible α L_α is closed under F, and Φ defines $F \cap L_\alpha$ in L_α. Many basic functions of set theory have this property, including ordinal exponentiation and various metamathematical functions; we will say that F is Σ_1^{KP}.

Define an ordinal α to be stable if L_α is a Σ_1-elementary substructure of L. If α is stable then α is admissible; but the converse is false. A cardinal κ is stable. To see this, recall that Σ_1-elementary substructures are closed under unions of chains, so it suffices to show that if $\beta < \kappa$ then there is an α with $\beta < \alpha < \kappa$ which is stable. This follows by the existence of a uniform Σ_1 Skolem function and the "condensation lemma" characterizing the transitive collapse (of the Skolem hull).

There is a Σ_1 predicate $x <_L y$ on L, which is a well-order on the class L. If $\beta < \alpha$ then the elements of L_β precede the elements of L_α in this order. If α is admissible, the formula defining $<_L$ defines its restriction to L_α in L_α. The function $F : \mathrm{Ord} \mapsto L$ which enumerates L in order of $<_L$ is Σ_1^{KP}.

There is a well-known alternative enumeration of L, which will be denoted $F_f : \mathrm{Ord} \mapsto L$. Unlike F, this enumeration is not bijective. It is due to Gödel; a thorough presentation can be found in Takeuti and Zaring [22]. The function F_f is Σ_1^{KP}. In fact $F_f[\alpha] = L_\alpha$ when α is an ϵ-number; this is shown in Linden [15], and it also follows that $F_f[\alpha]$ is amenable when α is a δ number.

Reference will be made below to some functions involved in the definition of F_f; these are denoted J_0, J, K_1, K_2, and K_3, as in Takeuti and Zaring [22]. J_0 is the standard pairing function on ordinals. J is a bijection from $\mathrm{Ord} \times \mathrm{Ord} \times 9$ to Ord, and K_i extracts the ith component of the inverse. These functions are Σ_1^{KP}, and in fact α is closed under them if it is a δ-number (see Linden [15]).

10. Recursion in an ordinal.

The definition of admissible ordinals reduced recursion theory in an ordinal α to recursion theory in L_α, by means of the map F. The latter is less cumbersome, and interest in recursion in an ordinal waned; however it will be useful in this paper. The paper Fukuyama [9] contains proofs of equivalence of some characterizations of recursion in an ordinal. We give a self-contained proof of an equivalence here, which suits the purposes of the paper.

For a structure S let Σ_1-Def(S) denote the relations definable in S by a Σ_1 formula with parameters. Let Π_1-Def(S) denote the complements of the relations in Σ_1-Def(S), and let Δ_1-Def(S) denote Σ_1-Def$(S) \cap \Pi_1$-Def(S). Clearly if Σ_1-Def$(S) = \Sigma_1$-Def(S') then Π_1-Def$(S) = \Pi_1$-Def(S') and Δ_1-Def$(S) = \Delta_1$-Def(S') as well. It is well known that certain types of relations in Σ_1-Def(S) are automatically in Π_1-Def(S), in particular functions and linear orders.

The structure α is too simple for carrying out recursion theory, and must be expanded. Indeed, ordinary recursion theory is carried out in ω, expanded with addition and multiplication.

Let K^s_α denote α, considered as a structure for the two-sorted language

$$\langle 0,\, 1,\, +,\, <,\, \mathrm{Len},\, \mathrm{Elem},\, \mathrm{Subseq}\rangle.$$

The first sort is the ordinals; 0, 1, +, and $<$ are as usual. Objects of the second sort are functions $s : \beta \mapsto \alpha$ for some $\beta < \alpha$; $\mathrm{Len}(s) = \beta$, the length of sequence s, $\mathrm{Elem}(s, \gamma) = s(\gamma)$ for $\gamma < \beta$ and 0 otherwise, and and $\mathrm{Subseq}(s, \beta, \gamma) = t$ iff $\mathrm{Elem}(t, \delta) = \mathrm{Elem}(s, \beta + \delta)$ for all $\delta < \gamma$. To complete the specification of K^s_α it remains to specify the allowed sequences; these will be the sequences which are in L_α. Bounded quantifiers on K^s_α are those of the form $\forall\gamma<\beta$ or $\exists\gamma<\beta$ (β can be a term). As a notational convenience, in this section Len(s) will be written as $|s|$ and Elem(s,β) as $s(\beta)$.

The structure K^f_α will also be considered, which adds to α the the predicate ϵ_f, where $\beta\in_f\gamma$ iff $F_f(\beta)\in F_f(\gamma)$; $\beta<\gamma$ abbreviates $\beta\in\gamma$.

Lemma 23. *Suppose α is an admissible ordinal.*

a. *The standard pairing function J_0 on ordinals is Δ_1 on K^s_α.*

b. *The Δ_1 predicates on K^s_α are closed under bounded quantification and substitution of Δ_1 functions.*

c. *The predicate ϵ_f is Δ_1 on K^s_α.*

Proof. For part a, let $R_0(\beta_1,\beta_2,\beta'_1,\beta'_2)$ be the predicate which specifies the usual well order on the ordered pairs $\langle\beta_1,\beta_2\rangle$ and $\langle\beta'_1,\beta'_2\rangle$; this is quantifier-free in the language with only $<$. Using R_0, define a predicate $P_1(s,t)$ which is true iff $|s|=|t|$ and $s(\gamma)=\gamma_1$ and $t(\gamma)=\gamma_2$ where $j_0(\gamma_1,\gamma_2)=\gamma$ for all $\gamma<|s|$. Let P_2 be $s(\beta)=\beta_1\wedge t(\beta)=\beta_2$. Then $J_0(\beta_1,\beta_2)=\beta$ iff $\exists s,t(P_1(s,t)\wedge P_2)$ iff $\forall s,t(P_1(s,t)\Rightarrow P_2)$. Further, s,t can be taken in L_α. For part b, we give two cases of the first claim, leaving the remainder of the proof to the reader. $\forall\gamma<\beta\exists\delta R(\gamma,\delta)$ can be rewritten as $\exists t(|t|=\beta\wedge\forall\gamma<\beta R(\gamma,t(\gamma)))$. $\forall\gamma<\beta\exists sR(\gamma,s)$ can be rewritten as $\exists t,u,v(|t|=\beta\wedge P_1(t,u)\wedge\forall\gamma<\beta R(\gamma,\mathrm{Subseq}(v,t(\gamma),u(\gamma))))$, where $P_1(t,u)$ states that $|t|=|u|$ and for all $\gamma<|t|$, $t(\gamma)$ is the sum of the $u(\delta)$ for $\delta<\gamma$. For the proof of part c, $s(\beta_1,\beta_2)$ will be written for $s(J_0(\beta_1,\beta_2))$, and similarly for t. The main step of the proof is to define a predicate $P_1(s,t)$ which is true if $|s|=|t|$; and for all β_1,β_2 with $J_0(\beta_1,\beta_2)<|s|$, $s(\beta_1,\beta_2)=1$ if $\beta_1<\beta_2$ and $F_f(\beta_1)\in F_f(\beta_2)$, else 0; and also $t(\beta_1,\beta_2)=1$ if $F_f(\beta_1)=F_f(\beta_2)$, else 0. The definition of P_1 can be given straightforwardly using Theorem 15.14 of Takeuti and Zaring [22]. Let

1. $P_2(\gamma,\beta)$ iff $\exists\gamma<\beta(s(\delta,\gamma)=1\wedge t(\gamma,\delta)=1)$;
2. $P_3(\beta_1,\beta_2)$ iff $\forall\gamma_1<\beta_1(s(\gamma_1,\beta_1)=1\Rightarrow\exists\gamma_2<\beta_2(P_2(\gamma_2,\beta_2)))$;
3. $F_1(\delta_1,\delta_2)=J(\delta_1,J(\delta_1,\delta_2,1),1)$; and
4. $F_2(\delta_1,\delta_2,\delta_3)=F_1(\delta_1,F_1(\delta_2,\delta_3))$.

The argument t of P_1 must satisfy $t(\beta_1,\beta_2)=1$ if $P_3(\beta_1,\beta_2)\wedge P_3(\beta_2,\beta_1)$, else $t(\beta_1,\beta_2)=0$. The conditions for $s(\gamma,\beta)=1$ (0 otherwise) break into nine cases, according to the value of $K_3(\beta)$; let $\beta_i=K_i(\beta)$ for $i=1,2$.

case 0: $\gamma<\beta$

case 1: $t(\gamma,\beta_1)=1\wedge t(\gamma,\beta_2)=1$

The remaining cases have the conjunct $s(\gamma,\beta_1)=1$, and the following.

case 2: $\exists\delta_1,\delta_2<\gamma(\gamma=F_1(\delta_1,\delta_2)\wedge P_2(\delta_1,\delta_2))$

case 3: $\forall\delta<\beta_2(t(\gamma,\delta)=1\Rightarrow s(\delta,\beta_2)=0)$

case 4: $\exists\delta_1,\delta_2<\gamma(\gamma=F_1(\delta_1,\delta_2)\wedge s(\delta_1,\beta_2)=1)$

case 5: $\exists\delta<\beta_2(P_2(F_1(\gamma,\delta),\beta_2))$

case 6: $\exists\delta_1,\delta_2<\gamma(\gamma=F_1(\delta_1,\delta_2)\wedge P_2(F_1(\delta_2,\delta_1),\beta_2))$

case 7: $\exists\delta_1,\delta_2,\delta_3<\gamma(\gamma=F_2(\delta_1,\delta_2,\delta_3)\wedge P_2(F_2(\delta_2,\delta_3,\delta_1),\beta_2))$

case 8: $\exists\delta_1,\delta_2,\delta_3<\gamma(\gamma=F_2(\delta_1,\delta_2,\delta_3)\wedge P_2(F_2(\delta_1,\delta_3,\delta_2),\beta_2))$

Letting $P_4(\beta_1,\beta_2)$ be "$\exists\gamma<\beta_2(s(\gamma,\beta_2)\wedge t(\gamma,\beta_1))$", $\beta_1\in_f\beta_2$ iff $\exists s,t(P_1(s,t)\wedge(P_4(\beta_1,\beta_2)\vee P_4(\beta_2,\beta_1)))$ iff $\forall s,t(P_1(s,t)\Rightarrow(P_4(\beta_1,\beta_2)\vee P_4(\beta_2,\beta_1)))$.

Theorem 24. *Suppose α is an admissible ordinal.*

a. *If $R(\beta_1, \ldots, \beta_n)$ is Δ_0 on K^s_α then it is Δ_1 on L_α.*

b. *If $R(\beta_1, \ldots, \beta_n)$ is Δ_0 on L_α then it is Δ_1 on K^f_α.*

c. *If $R(\beta_1, \ldots, \beta_n)$ is Δ_0 on K^f_α then it is Δ_1 on K^s_α.*

Proof. For part a, the symbols of K^s_α are readily verified to be Δ_1 on L_α. The bounded quantifier $\forall\gamma < \beta_i$ translates to $\forall\gamma \in \beta_i$ (actually an abbreviation), and similarly for $\exists\gamma \in \beta_i$. For part b, we first show that if $S(x_1, \ldots, x_n)$ is Δ_0 on L_α then $S(F_f(\beta_1), \ldots, F_f(\beta_n))$ is Δ_1 on K^f_α. This is clear for the atomic formula $x_i \in x_j$. If Φ is $\forall x \in x_i$, the translation Φ' of Φ is $\forall\gamma < \beta_i(\gamma \in_f \beta_i \Rightarrow \Psi')$. To complete the proof it suffices to find a Δ_1 S such that $R(\beta_1, \ldots, \beta_n)$ iff $S(F_f(\beta_1), \ldots, F_f(\beta_n))$; let G be a Δ_1 function choosing a preimage of F_f, and let S be $S(G(\beta_1), \ldots, G(\beta_n))$. For part c, the symbols of K^f_α are Δ_1 on K^s_α by Lemma 23. The bounded quantifier $\forall\gamma < \beta_i$ translates to itself, and similarly for $\exists\gamma \in \beta_i$.

Further consequences are readily proved; for example, if $R(\beta_1, \ldots, \beta_n)$ is a Δ_0 relation on K^s_α then the $R(F^{-1}(x_1), \ldots, F^{-1}(x_n))$ is a Δ_1 relation on L_α. There is a variant of K^s_α, where the second sort is omitted in favor of a predicate for the ordinals which code sequences. To use any of these structures, similar facts must be proved, and although the most artificial, K^f_α has properties which make it most convenient for the sequel.

In particular recursion theory on K^f_α is readily defined, since definitions can be given in L_α by Lemma 24b. The Σ_1 formulas with parameters in α and exactly one free variable can be coded as ordinals less than α. Let $\mathrm{Tru}(\phi, \xi)$ be the predicate which is true iff the ordinal ϕ is such a code, and the formula is true when the value ξ is assigned to the free variable. There is a Σ_1 formula in the language of K^f_α which defines this predicate in any K^f_α (noting that the code of a formula of $K^f_{\alpha'}$ with $\alpha' < \alpha$ is the same as its code in α).

In recursion theoretic terms, ϕ is a code for the recursively enumerable unary predicate defined by the formula coded by ϕ, which will be denoted W_ϕ. Multivariate predicates can be defined either similarly, or using a pairing function such as J_0. An enumeration of the partial recursive functions may be obtained by "single value-izing" the binary predicates in a well-known manner; let P_ϕ denote this enumeration. The recursion theorem is readily proved.

Lemma 25. *For any ϕ there is a θ such that $P_\theta(\xi) = P_\phi(\langle\theta, \xi\rangle)$ for all ξ.*

Proof. Let f be an α-recursive total function such that $P_{f(\phi)}(\xi) = P_\phi(\langle\phi, \xi\rangle)$. Let π be such that $P_\pi(\langle\rho, \xi\rangle) = P_\phi(\langle f(\rho), \xi\rangle)$. Let $\theta = f(\pi)$.

It is not necessary to single value-ize to obtain a recursion theorem. For any ϕ there is a θ such that $W_\theta(\langle\xi, \tau\rangle) = W_\phi(\langle\langle\theta, \xi\rangle, \tau\rangle)$; the proof is virtually identical.

11. Constructive ordinals in α.

Let α be an admissible ordinal and let P_ϕ for $\phi < \alpha$ be an enumeration of the partial recursive functions from α to α, i.e., those which are Σ_1, in either L_α or K^f_α. Let W_ϕ be an enumeration of the recursively enumerable (Σ_1) subsets. Recursion theory in an arbitrary admissible ordinal is more complex than recursion theory in ω, but simpler than recursion theory in an arbitrary admissible set. Included in this section is a review of some basic facts which hold in an admissible ordinal, useful to the purposes of the paper.

Care must be taken in adapting basic facts; for example it is not true that a bounded recursively enumerable subset is in L_α. It is true that a nonempty recursively enumerable set S is the range of a total recursive function f; if S is defined by $\exists\gamma\Phi(\gamma, \beta)$ then $f(\langle\gamma, \beta\rangle)$ equals β if Φ is true, else some fixed member of S.

Certain ordinals Ω_β (not necessarily in α) can be coded by certain ordinals β (in α), in an effective manner. These will be called the constructive ordinals. The constructive ordinals in ω were first investigated by Church and Kleene in the late 1930's, and have been extensively studied since. In the mid 1960's it was shown that their supremum, which is denoted ω_1^{CK}, is the next admissible ordinal after ω.

There is a predicate $<_O$, with field O, which is the least predicate satisfying the following conditions.

1. $0 <_O 1$.

2. If $\beta \in O$ then $\beta <_O \beta \cdot 3 + 1$.

3. If P_β is total, increasing, and $P_\beta[\alpha] \subseteq O$, then for all $\gamma < \alpha$ $P_\beta(\gamma) <_O \beta \cdot 3 + 2$.

4. If $\delta \in \mathrm{Lim} \cap \alpha$, P_β is defined and increasing on δ, and $P_\beta[\delta] \subseteq O$, then for all $\gamma < \delta$, $P_\beta(\gamma) <_O \langle\beta, \delta\rangle \cdot 3 + 3$.

5. $<_O$ is transitive.

Clause 4 is not present ordinary recursion theory, where it is vacuous. Little more than the definition of O is required for the sequel, but some basic facts will be stated, in many cases leaving it to the reader to verify that the proofs in the case of ω of Sacks [19] carry over.

The predicate $<_O$ is Π^1_1 (uniformly), and is a well-founded partial order; O is Π^1_1 (uniformly). $\beta \cdot 3+1 \in O$ iff $\beta \in O$, and in this case there is no ζ with $\beta <_O \zeta <_O \beta \cdot 3+1$. $\beta \cdot 3+2 \in O$ iff P_β satisfies the conditions in clause 3 above; and in this case there is no ζ with $\zeta <_O \beta \cdot 3+2$ and $P_\beta(\gamma) <_O \zeta$ for all γ. $\langle \beta, \delta\rangle \cdot 3+3 \in O$ iff P_β and δ satisfy the conditions in clause 4; and in this case there is no ζ with $\zeta <_O \langle\beta,\delta\rangle \cdot 3+3$ and $P_\beta(\gamma) <_O \zeta$ for all $\gamma < \delta$.

If $\beta \in O$ then $\{\gamma : \gamma <_O \beta\}$ is linearly ordered by $<_O$. Furthermore, there are total recursive functions p and q such that $W_{p(\beta)} = \{\gamma : \gamma <_O \beta\}$, and $W_{q(\beta)} = \{\langle\gamma_1,\gamma_2\rangle : \gamma_1 <_O \gamma_2 <_O \beta\}$. For $\beta \in O$ let $\Omega_0 = 0$, $\Omega_{\beta\cdot 3+1} = \Omega_\beta + 1$, $\Omega_{\beta\cdot 3+2} = \sup\{\Omega_{P_\beta(\gamma)}\}$, and $\Omega_{\langle\beta,\delta\rangle\cdot 3+3} = \sup\{\Omega_{P_\beta(\gamma)} : \gamma < \delta\}$.

There is a recursive function $+_O$ such that $\beta, \gamma \in O$ iff $\beta +_O \gamma \in O$, and in this case $\Omega_{\beta+\gamma} = \Omega_\beta + \Omega_\gamma$. Further

- if $\beta, \gamma \in O$ and $\gamma \neq 0$ then $\beta <_O \beta +_O \gamma$;
- if $\beta \in O$ and $\beta <_O \delta$ then $\beta +_O \gamma <_O \beta +_O \delta$; and
- $\beta, \gamma, \delta \in O$ and $\gamma = \delta$ iff $\beta +_O \gamma = \beta +_O \delta$.

The recursion equations satisfied by $+_O$ are the following.

$$\begin{aligned}
\beta +_O 0 &= \beta \\
\beta +_O (\gamma \cdot 3+1) &= (\beta +_O \gamma)\cdot 3+1 \\
\beta +_O (\gamma \cdot 3+2) &= \gamma' \cdot 3+2 \text{ where } P_{\gamma'}(\zeta) = \beta +_O P_\gamma(\zeta) \\
\beta +_O (\langle\gamma,\delta\rangle \cdot 3+3) &= \langle\gamma',\delta\rangle \cdot 3+3 \text{ where } P_{\gamma'}(\zeta) = \beta +_O P_\gamma(\zeta) \text{ for } \zeta < \delta, \\
&\quad \text{and is undefined for } \zeta \geq \delta
\end{aligned}$$

An index for such a function can be obtained as in the case of ordinary recursion theory. This is an example of a definition by by "ETR" (effective transfinite recursion).

There is a recursive function g such that for all ϕ, $g(\phi) \in O$ iff $W_\phi \subseteq O$; and in this case $\Omega_\beta < \Omega_{g(\phi)}$. As in the case of ordinary recursion $g(\phi)$ is obtained by taking the sum of the elements of W_ϕ.

If R is a well-founded binary relation let $\mathrm{Ht}(R)$ denote its height. There is a recursive function f such that the relation $R(\beta,\gamma) = W_\phi(\langle\beta,\gamma\rangle)$ is well-founded iff $f(\phi) \in O$; and in this case $\mathrm{Ht}(R) \leq f(\phi)$. This may be defined by a recursion, using the function g of the preceding paragraph. (The proof of Lemma I.4.3 in Sacks [19] seems incomplete. The recursion lemma used in the proof of Theorem 16.XXI of Rogers [18] can be used to complete the proof, and it holds in any α.)

Let r denote a binary relation on α, which is a well-order of its field; let $\mathrm{Ot}(r)$ denote its order type. Say that an ordinal β is constructive (over α) if is has a notation in O. Consider the following statements.

1. β is constructive.
2. $\beta = \mathrm{Ot}(r)$ where r is Σ_1 on α.
3. $\beta = \mathrm{Ot}(r)$ where r is Δ_1 on α.

Using the preceding facts, it may be seen that conditions 1 and 2 are equivalent. Indeed, if β is constructive then β is the order type of the relation $W_{q(\beta)}$ described above. Conversely if condition 2 holds, then r is a recursively enumerable well-founded binary relation on α, so its height, which equals β, is a constructive ordinal by the above.

The conditions 2 and 3 on r are distinct. Indeed, if r is a linear order on its field, and is recursively enumerable, then r is recursive iff its field f is recursive. For one direction, $\neg r(x,y)$ iff $\neg f(x) \vee \neg f(y) \vee x = y \vee r(y,x)$. For the other, suppose f has cardinality at least 2; then $f(x)$ iff $\exists y\, r(x,y) \wedge \exists y\, r(y,x)$, and $\neg f(x)$ iff $\exists y \exists z (r(y,z) \wedge x \neq y \wedge \neg r(x,y) \wedge \neg r(y,x))$. Even though the conditions are distinct, the order types are the same. This follows because there is a recursive function g mapping some ordinal $\delta \leq \alpha$ bijectively to the field of r; and the relation $r(g(x), g(y))$ has the same order type as r.

To see that the function g of the preceding paragraph exists, let $\exists w\, s(w,\beta)$ be a set where s is recursive. Define $g(\xi) = \gamma$ iff there exists a ζ such that in the first ζ pairs in a suitable enumeration of $L_\alpha \times \alpha$,exactly $\xi+1$ values of β occur for which $s(w,\beta)$ holds for some w; and further γ is that β where the first occurrence with $s(w,\beta)$ is last. As a further observation, if the set is unbounded then the domain of g is α (of course, if α is projectible then smaller domains suffice).

Let η denote the next largest admissible ordinal after α. Other properties that an ordinal β might have include the following, where r is as above.

4. $\beta = \mathrm{Ot}(r)$ where r is Δ_1 on L_η.

5. $\beta = \mathrm{Ot}(r)$ where $r \in L_\eta$.

6. $\beta < \eta$.

By Δ_1-separation in L_η 4 implies 5, and 5 implies 4 trivially. The equivalence of 5 and 6 follows by Theorem V.5.9 of Barwise [1].

Let α_C denote the supremum of the constructive ordinals. Clearly $\alpha_C \leq \eta$; $\alpha_C < \eta$ can hold, indeed it suffices that α be a cardinal of uncountable cofinality. A proof of this follows.

Consider the predicate "Φ is a Σ_1 formula with parameters from L_α, r is a binary relation whose field is contained in L_α, Φ defines r, and r is a well-order on its field". If this predicate (call it Σ_1-WO) is Δ_1 on L_η, a function which is Σ_1 on L_η can be defined, whose domain is contained in L_α and whose range is the order types, from which it follows that $\alpha_C < \eta$. Σ_1-WO is always Σ_1 on L_η, so it suffices that it be Π_1.

Let $f(\alpha)$ be the least admissible ordinal β such that $\beta > \alpha$ and L_β is admissible and satisfies Σ_1 separation. Let D_α be the set of functions from ω to α which are elements of $L_{f(\alpha)}$. Using Theorem I.9.6 of Barwise [1] it follows that for $\alpha > \omega$, if a partial order $r \subseteq \alpha \times \alpha$ is Σ_1 on L_η and is not well-founded, then there is an $f \in D_\alpha$ whose range is a descending chain in r. Thus if $D_\alpha \cap L_\eta \neq \emptyset$ then Σ_1-WO is Π_1 on L_η. If α is a cardinal of uncountable cofinality then in fact $D_\alpha \subseteq L_\alpha$.

12. Interpreting $K^f_{\kappa^+}$ in classes.

The Σ_1 formulas of $K^f_{\kappa^+}$ may be interpreted as Σ^1_1 formulas of set theory, in such a way that recursion theoretic arguments in the former setting can be carried out in the latter. There is a Δ^1_0 predicate $\mathrm{OC}(X)$ stating that the class X is the code of an ordinal $\alpha < \kappa^+$, that is, a well-order. This states that X is a class of ordered pairs, which as a binary relation is transitive and reflexive, total, and has no descending chains of length ω. The formula defines the desired class of codes, in any V_κ where $\kappa \in \mathrm{Inac}$.

Bold Greek symbols $\boldsymbol{\alpha}, \boldsymbol{\beta}, \ldots$ will be used to denote second order values restricted to satisfy OC. The usual order predicate may be defined on these values in a Δ^1_1 and uniform (in any V_κ where $\kappa \in \mathrm{Inac}$) manner. Indeed, there is a Δ^1_0 predicate stating that F is a function witnessing $\boldsymbol{\alpha} < \boldsymbol{\beta}$, etc. The value of the predicate $\boldsymbol{\alpha} < \boldsymbol{\beta}$ depends only on the ordinals coded by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, and not the particular well-orders.

The predicate $\boldsymbol{\alpha} \in_f \boldsymbol{\beta}$ may be similarly defined, by recasting the proof of Lemma 23 using classes for the sequences. A bounded sequence of ordinals is coded as a class of triples $\langle \xi, \langle x, y \rangle \rangle$ where ξ is an ordinal. The binary relation R_ξ obtained by fixing ξ is a well-order for all ξ; R_0 is the domain of the sequence, and $R_{1+\xi}$ is $\boldsymbol{\alpha}_{\boldsymbol{\xi}}$ where $\boldsymbol{\xi}$ is coded by ξ in R_0. The predicate defining these codes is Δ^1_0. The predicates required to carry through the recasting of Lemma 23 are all Δ^1_1 (and in some cases Δ^1_0).

Let $\boldsymbol{O}$ denote the classes of V_κ coding members of O in L_{κ^+}. Such classes code ordinals less than κ^+_C. To define them, it suffices to replace ordinals by boldface ordinals in the closure conditions used to define $<_O$; each boldface ordinal may be any class representing it. For $\boldsymbol{\beta} \in \boldsymbol{O}$, the ordinal $\Omega_{\boldsymbol{\beta}}$ is defined to be Ω_β where β is the member of O in L_{κ^+} which $\boldsymbol{\beta}$ represents.

13. R-superschemes.

An R-superscheme in κ is defined to be a class $\boldsymbol{\Sigma}$ of V_κ in the family $\boldsymbol{O}$, satisfying certain restrictions (given in the next paragraph). Strictly speaking $\boldsymbol{\Sigma}$ is not a superscheme; rather it is a well-founded tree. This issue will not be considered further here, although questions additional to those raised in Section 7 arise. The "length" of the "superscheme" is $\Omega_{\boldsymbol{\Sigma}} + 1$, and $\Omega_{\boldsymbol{\Sigma}}$ may be a limit ordinal; the notion of X_α is replaced by $X_{\boldsymbol{\Sigma}}$.

Note that $\mathrm{Cf}(\Omega_\beta) = \mathrm{Cf}(\beta)$. The restrictions on $\boldsymbol{\Sigma}$ may thus be given as follows. If $\boldsymbol{\Sigma}$ is of the form $\langle \boldsymbol{\beta}, \boldsymbol{\delta} \rangle \cdot 3 + 3$, then the order type of the well-order $\boldsymbol{\delta}$ is required to be either κ, or less than κ. If $\boldsymbol{\Sigma}$ is of the form $\boldsymbol{\beta} \cdot 3 + 2$, $\mathrm{Cf}(\Omega_{\boldsymbol{\Sigma}})$ is κ^+.

In definitions by ETR on R-superschemes, the recursion may be broken up into the 5 cases of the definition, namely

1. $\alpha = 0$,

2. $\alpha = \beta + 1$,

3. $\alpha = \sup\{\alpha_\xi : \xi < \delta\}$ where $\delta < \kappa$,

4. $\alpha = \sup\{\alpha_\xi : \xi < \kappa\}$, and

5. $\alpha = \sup\{\alpha_\xi : \xi < \kappa^+\}$.

Indeed, the necessary predicates on the codes are Δ^1_1 in V_κ, a fact whose verification is facilitated by making use if Σ_1 definability in $K^f_{\kappa+}$ and its interpretation.

Recall the predicate Tru of Section 2; the notation will be abused by writing $\mathrm{Tru}(\Phi)$ for a sentence Φ with a class parameter. The predicate $\models_{V_\lambda} \Phi$ for a sentence Φ will also be required.

A Π^1_1 formula $\Psi_{\boldsymbol{\Sigma}}(\lambda)$ intended to define $\mathrm{H}_{\boldsymbol{\Sigma}}(\mathrm{Inac}_\kappa)$ in V_κ will be defined by ETR. In case 1 this is just the (first order) definition of Inac, namely,

$$\omega \in \lambda \wedge \forall x \in V_\lambda \exists y \in V_\lambda (y = P(x)) \wedge$$
$$\forall F \subseteq V_\lambda \forall x \in V_\lambda (F \text{ a function} \Rightarrow \exists y \in V_\lambda (y = F[x])).$$

In case 2 it is

$$\Psi_{\boldsymbol{\Sigma}}(\lambda) \wedge \forall Y \subseteq \lambda (Y \text{ club} \Rightarrow \exists \mu (\mu \in Y \wedge \Psi_{\boldsymbol{\Sigma}}(\mu))).$$

In case 3 it is

$$\lambda \in \mathrm{Inac} \wedge \forall \xi < \delta \mathrm{Tru}(\Psi_{\boldsymbol{\Sigma}_\xi}, \lambda))$$

(with obvious notation; $\lambda \in \mathrm{Inac}$ can be replaced by weaker requirements). Case 4 replaces δ by κ. In case 5 it is

$$\lambda \in \mathrm{Inac} \wedge \forall \xi \models_{V_\lambda} \mathrm{Tru}(\exists \mu \Psi_{\boldsymbol{\Sigma}_\xi}(\mu)).$$

A Π^1_1 sentence $\Phi_{\boldsymbol{\Sigma}}$ intended to enforce $\mathrm{H}_{\boldsymbol{\Sigma}}(\mathrm{Inac}_\kappa)$ will be defined by ETR. The first 4 cases are similar to formulas in Section 2; in fact case 1 is identical, namely,

$$\exists x (x = \omega) \wedge \forall x \exists y (y = P(x)) \wedge \forall F \forall x (F \text{ a function} \Rightarrow \exists y (y = F[x])).$$

For case 2, the sentence is

$$\Phi_{\boldsymbol{\Sigma}} \wedge \forall Y (Y \text{ club} \Rightarrow \exists \mu (\mu \in Y \wedge \models_{V_\mu} \Phi_{\boldsymbol{\Sigma}})).$$

For case 3 it is

$$\Phi_0 \wedge \exists x (x = \delta) \wedge \forall \xi < \delta \mathrm{Tru}(\Phi_{\boldsymbol{\Sigma}_\xi}).$$

Case 4 replaces δ by κ and drops the existence of δ. In case 5 it is

$$\Phi_0 \wedge \forall \xi \mathrm{Tru}(\exists \mu \Psi_{\boldsymbol{\Sigma}_\xi}(\mu)).$$

Theorem 25. *Suppose κ is Π^1_1-indescribable.*

a. $\models_{V_\kappa} \Phi_{\boldsymbol{\Sigma}}$.

b. *If* $\models_{V_\lambda} \Phi_{\boldsymbol{\Sigma}}$ *then* $\models_{V_\kappa} \Psi(\lambda)$.

Proof. The proof is by induction. All cases of part b are immediate. The first 4 cases of part a are similar to results already proved and left to the reader. For case 5, by induction and the properties of κ, for all ξ $\Phi_{\boldsymbol{\Sigma}_\xi}$ holds in some V_λ, and the claim follows by part b.

The foregoing suggests that by attending to various details, it can be shown that for a weakly compact cardinal κ, $\rho(\kappa) \geq \kappa^+_C$. However, the remarks at the end of Section 12 suggest that $\rho(\kappa) > \kappa^+_C$, indeed that $\{\lambda < \kappa : \rho(\lambda) \geq \lambda^+_C\}$ is in the enforceable filter.

14. Constructibility.

For the following theorem, recall that if $\kappa \in \mathrm{Inac}$ then $(\kappa \in \mathrm{Inac})^L$.

Theorem 26. *Given $\kappa \in \mathit{Inac}$, and $\Sigma \in \mathcal{S}^\kappa_\sigma \cap (\mathcal{S}^\kappa_\sigma)^L$, $H_\Sigma(\mathit{Inac}_\kappa) \subseteq (H_\Sigma(\mathit{Inac}_\kappa))^L$.*

Proof. The statement that κ is weakly inaccessible is Π_1, so if true is true in L; further in L if κ is weakly inaccessible then it is inaccessible. This proves the theorem for $\alpha = 0$. For the case $\alpha + 1$, if $\lambda \in X_{\alpha+1}$ (where X is used for Inac_κ) then by induction $\lambda \in X^L_\alpha$. By hypothesis X_α is stationary below λ, so by induction X^L_α is. Now, "X is stationary below λ" is Π_1, so X^L_α is stationary below λ in L, that is, $\lambda \in X^L_{\alpha+1}$. The cases of intersection and diagonal intersection are left to the reader.

Theorem 27. *If κ is greatly Mahlo then this is true in L.*

Proof. We leave it to the reader to show that κ is greatly Mahlo iff $\mathrm{H}_\Sigma(\mathrm{Inac}_\kappa) \neq \emptyset$ for any prescheme Σ in κ; that Theorem 26 holds for preschemes; and that the notion of prescheme is absolute. The theorem follows.

Whether further such statements can be made should be investigated. One suspects that in any case, cardinals which have been built up according to adequate standards will not contradict $V = L$. This is the case for weakly compact cardinals for example (see Jech [12]). This in turn is cause to suspect that cardinals contradicting $V = L$ do not exist, and indeed that $V = L$. Hopefully, as research in this area proceeds, more quantitative statements will be discovered.

The remainder of this section will be concerned with other arguments in favor of $V = L$; some of these have been given in Dowd [5]. In trying to produce arguments deciding independent questions, mathematics becomes an empirical science, attempting to adduce properties of mathematical reality. Any decisions must be agreed on by a substantial portion of the mathematical community to become the prevailing view. The acceptance of ZFC shows that such agreement is possible, and indeed that mathematics does have this empirical aspect.

To resolve independent questions it is in fact necessary to give empirical arguments. A variety of such which seems compelling to the author is that certain facts would be apparent if they were true. Certain objects whose existence has not been demonstrated (indeed is independent) can be argued to have such a character that constructions would exist if the objects did. ZFC gives us a sufficiently clear picture of mathematical reality that these constructions could be proved to result in sets with the required properties. Examples will be given below. Admittedly, there must be much debate before any conclusions can be reached, and the remarks here present only one point of view, among what will surely be many.

Our first argument in favor of $V = L$ is that L is a standard transitive model of ZFC containing Ord. ZFC exhausts the principles (other than collecting the universe) of obtaining sets, so the fact that L is a natural model of the axioms suggests that it is all the sets.

A second argument on general principles is that $V = L$ is a "master" nonconstructive existence principle, which settles many independent questions because it supplies a spectrum of progressively deeper such principles. This can be seen as the most reasonable behavior of sets, in comparison to principles contradicting $V = L$. For example, a strong choice function follows, and the classical position would be that such exists. For another, the axiom of choice suffices to produce an undetermined game; with $V = L$ a projective such can be constructed.

The continuum hypothesis (CH) is crucial to these considerations. It is the next simplest nonconstructive existence principle after the axiom of choice (AC), stating the existence of a surjection from $\aleph_1$ to $P(\omega)$. Although there has been debate on the subject, AC is generally considered to be clearly true (an argument in terms of "stages" for example can be found in Shoenfield [20]). It is used throughout mathematics, including set theory, resulting for example in a more convincing theory of cardinality than is obtained without AC.

AC is often considered to be more suspect than the other axioms of set theory, due to the fact that unlike the other existence axioms, the defining property of a choice function does not uniquely specify the function. The assumption that the function exists is "nonconstructive"; but the function clearly does exist, showing that nonconstructive existence principles are necessary in set theory.

The truth of CH is a more subtle question. CH is "higher than" AC, in that it is less obvious, and asserts the existence of a more specific function. Nevertheless, arguments can be given in its favor. To decide CH, one must choose between the existence of a surjection from $\aleph_1$ to $P(\omega)$, and an injection from $\aleph_2$. But it seems clear that if the injection existed this would be apparent. If it existed, it would be an object of such mathematical importance that various structures in topology and analysis would exist, which Cantor would have discovered in his application of aggregate theory to these areas, if not Lebesgue. It could not be hidden behind the need for nonconstructive existence principles, and would have a straightforward construction. The surjection, however, is exactly the type of object which is hidden (unless, of course, we assume $V = L$).

The contrast between the proofs of the consistency of the two principles further illustrates the point. There is a function f which, if one makes the assumption that every set has some property (constructibility), is a surjection of $\aleph_1$ onto $P(\omega)$. On the other hand there is no known function which merely requires some such assumption to be an injection of $\aleph_2$ in $P(\omega)$. Indeed, this is an example of how arguments in favor of CH and of $V = L$ reinforce each other. Note also that Gödel's discovery of L rendered fruitless the search for an embedding of $\aleph_2$. Cohen's discovery of forcing, on the other hand, rendered fruitless the search for a "constructive" surjection of $\aleph_1$. Of course, there statements are merely empirical; but this is how the matter must be settled if it is to be.

Finally, the independence of the existence of an injection of $\aleph_2$ can be contrasted with the straightforward construction of injections of $\aleph_1$. Indeed, the latter argument "breaks down" when one tries to extend it to $\aleph_2$. This is what led the author to suspect, in 1985, that CH might be true.

Once one accepts that CH is a true nonconstructive existence principle, it is not so great a step to accept $V = L$. Firstly, $V = L$ is of the character of existence principle which seems to be true. It asserts the existence of a construction sequence for each set. There is no reason to reject this. The mere fact that L is a model of ZFC is evidence that L is all the sets. The intricate logical properties of construction sequences can then be seen as implying various more specific principles (CH being the second simplest).

Secondly, CH states that there is some enumeration of $P(\omega)$ by $\aleph_1$. $V = L$, up to $\aleph_1$, states that the constructibility process will produce the hereditarily countable sets by stage $\aleph_1$. That it will produce $P(\omega)$ is assured by CH, with the added proviso that not only is there some well-order, but one that satisfies reasonable conditions on its behavior. That it will produce the hereditarily countable sets then follows (see the remarks preceding Proposition VII.1.2 of Sacks [19]).

There are other arguments in favor of $V = L$. In order for $\omega_1^L < \omega_1$ to be true, the constructibility process would have to stop producing new maps from ω through a countable ordinal at some countable stage. This may be seen as unlikely. A countable ordinal is infinitesimally small compared to ω_1. The "homogeneity" of the constructibility process would have to break down at some point for which there must be an overwhelming reason for the breakdown. This purported reason suffers the fate of not existing if $V = L$.

It is well known that forcing arguments can be given to show that $\omega_1^L = \omega_1$ must be assumed. In addition, significantly to the observations made at the start of this section, the existence of measurable cardinals must be rejected. It is reasonable to take the position that the constructibility process cannot "get stuck", and so there is something "wrong" with measurable cardinals. Indeed, what is wrong with them is that they cannot be built up. The assumptions made about the filter the cardinal bears are arbitrary and too strong.

There has been argument that $V \neq L$, because large cardinal theory is a robust and highly developed branch of set theory. A counter-position is that, the independence of the strongest nonconstructive existence principles is attested to not only by forcing arguments, but also by axioms asserting the existence of "pathological" large cardinals. These axioms appear to be consistent, as do other axioms which involve filters in cardinals which have "pathological" properties.

15. Conclusion.

Theorem 25 probably does not exhaust the methods for iterating Mahlo's operation within a weakly compact cardinal; on the other hand it probably comes close to doing so. The machinery of superscheme theory has yielded this result, using recursion theoretic methods in the second order language of set theory over V_κ. Clearly the most important next step in this area of research is the establishment of lower and upper bounds on the superscheme rank of a weakly compact cardinal, and an unsurpassable lower bound might simply require more powerful recursion theoretic methods. An upper bound could possibly require the assumption $V = L$; in this case $V_\kappa = L_\kappa$ and many powerful recursion theoretic methods are available, although second order methods might need to be further developed.

Although further research should provide more definitive evidence, the results of this paper already suggest that weakly compact cardinals exist. Collecting the universe by effectively iterating Mahlo's operation builds up indescribable cardinals. Weakly compact cardinals are a "sound barrier", and once it is broken the existence of more highly indescribable cardinals seems to follow inevitably. Presumably their rank can be computed; this is of interest for example for the I-indescribable cardinals of Section 3.

Further research needs to be done on superschemes. In particular dispensing with them in favor of trees, and relaxing the restrictions, should be considered.

Suggestions as to further research remain more speculative, but clearly a central questions is,

- "Can $\kappa(\omega)$ be built up?"

This question is particularly important because it leads to the question of how indiscernables ("innocuous" as for $\kappa(\omega)$ or "pathological" as for 0#) relate to the filters which are shown to exist in building up cardinals.

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