A LOWER BOUND ON THE LENGTH OF BASIC MINIMAL 1-(3t+1,3) DESIGNS

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Abstract: In a previous paper the author found some minimal 1-(v,3) designs with large b, by exhaustive search. Here some further such are found, by more ad-hoc methods. A construction is given for v = 3t + 1 for t ≥ 2, where b grows quadratically with v.

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1. Introduction

Suppose v and k are integers with 1 ≤ k ≤ v. A 1-design with parameters v and k is a v row by b column matrix for some b, with elements either 0 or 1, such that each column contains k 1’s, and each row contains r 1’s for some r. Such a matrix is exactly the incidence matrix of a “biregular bipartite graph”, with v vertices of degree r in one class, and b vertices of degree k in the other class. The notation “1-(v,k) design will be used to denote such designs.

There is a well-known system of linear equations associated with 1-designs. Various facts of interest about this system are proved in [1]. The following further fact is of interest, both for the theory of 1-designs and for computations.
Theorem 1. Suppose $D$ is a basic minimal 1-design, and $S$ is the set of its columns. Then $D$ is determined by $S$.

Proof. Suppose first that $S$ is nondegenerate, i.e., $|S| = v$ where there are $v$ rows. Let $M$ be the matrix $[C_1,\ldots,C_v]$ where $S = \{C_1,\ldots,C_v\}$. Let $x$ be the column vector of indeterminates $x_{C_1},\ldots,x_{C_v}$. The system $Mx = J_{v \times 1}$ (see [1] for notation) has a unique solution $x$. $D$ is obtained when $x$ is converted to an integer vector with the integers having no common divisor. In general, suppose $T_1$ and $T_2$ are extensions of $S$ to bases. It is readily seen that in the primal simplex tableau there is a sequence of level 0 pivots from $T_1$ to $T_2$. Further, the solutions for $T_1$ and $T_2$ have the same restrictions to $\{x_C : C \in S\}$. See [5] for terminology.

2. Long designs

Theorem 6 and 7 of [1] give bounds on the length $b$ of minimal and basic minimal 1-$(v,k)$ designs. It is of interest to both design theory and to linear programming theory to obtain improvements to these bounds, and to obtain lower bounds. As seen in [1], these questions are of interest even for $k = 2$.

For $5 \leq v \leq 8$ the exact bound for basic minimal 1-designs was determined in [1] by exhaustive search using vertex enumeration. For $k = 3$ these values are $b=5$, $b=21$, and $b=48$ respectively. (There is an error in Table 1 of [1]; the number of non-isomorphic basic minimal 1-$(6,3)$ designs is 3, not 4).

In this paper, a lower bounds for basic minimal solutions with $k = 3$ will be given. Before giving this, a method for obtaining examples and lower bounds for small $v$ will be given, which uses random search rather than exhaustive search.

Using notation as in [1], let $M$ be the matrix with $v$ rows labeled $0,\ldots,v-1$ and $\binom{v}{k}$ columns, the $k$ element subsets of $\{0,\ldots,v-1\}$. Let $M^+_H$ be the matrix derived from $M$ by subtracting row 0 from the other rows and replacing it by a row of all 1’s. A vector $x$ is a solution to $Mx = J_{v \times 1}$ iff $y = (v/k)x$ is a solution to $M^+_H y = e_0$ where $e_s$ denotes the (column) unit vector with 1 in row $s$. It follows that the null spaces of $M$ and $M^+_H$ are the same, and so the sets of columns which are bases is the same (labeling a column of $M^+_H$ with the column of $M$ from which it is derived).
Solutions to the LP $M_H^T x = e_0, x_C \geq 0$ for all $C$ will be considered. This LP is in standard form, and is readily solved using the primal simplex method, as described in [5] for example. The two-stage method for finding an initial feasible basis may be avoided, if a feasible basis is known. The columns of it may be transformed into distinct unit vectors by an arbitrary sequence of pivots on the columns of the basis.

For $\gcd(v,k) = 1$ define the cyclic design to be that where column $i$ is $\{0+i \mod v, \ldots, k-1+i \mod v\}$.

**Theorem 2.** For $\gcd(v,k) = 1$ the cyclic design is basic.

*Proof.* Since $\gcd(v,k) = 1$, $1 + x + \cdots + x^{k-1}$ and $x^v - 1$ are relatively prime. The theorem follows by circulant matrix theory (see [3]).

**Theorem 3.** Suppose $\gcd(v,k) = 1$ and $v = qk + s$ where $q \geq 2$ and $0 < s < k$. Let $x$ be the vector where $x_C = k/v$ if $C$ has 1’s in rows $kt$ to $kt+k-1$ for some $t$ with $0 \leq t \leq q-2$; $x_C = 1/v$ if $C$ is one of the columns of the cyclic design in rows $k(t-1)$ to $v-1$; and $x_C = 0$ otherwise. Then $x$ is a solution to $M_H^T x = e_0$ which maximizes $x_{\{0,\ldots,k-1\}}$.

*Proof.* Let $B$ be the matrix where for $j < v-k$ column $j$ has 1’s in rows $j, \ldots, j+k-1$; and in the remaining columns there is a copy of the cyclic design in the lower right. $B$ is block upper triangular, where the blocks along the diagonal are invertible (using theorem 2), so $B$ is basic. Direct computation shows that $Bx = (k/v)J_{v \times 1}$. Direct computation also shows that the top row of $(B_H^+)^{-1}$ has $k/v$ in column 0 and $-1/v$ in the other columns. Let $c$ be the (row) cost vector, with $-1$ in column 0 and 0’s elsewhere. Let $c_B$ be the similarly defined vector of length $v$. As noted in section 2.6 of [5], the relative cost vector of the simplex tableau at the basis $B$ equals $c - c_B(B_H^+)^{-1}M_H^+$. Direct computation shows that this is 1 in column $C$ if $0 \in C$, except for column 0, which is 0; and 0 in the remaining columns.

The maximum value of $x_C$ for the cyclic design is $1/v$. Starting from this basis, the simplex algorithm may be run with random pivots. The basic feasible solution with the largest value of $b$ can be monitored. For $k = 3$, this was done for $7 \leq v \leq 20$ with
gcd(v, 3) = 1. For a given v, 100,000 repetitions were performed. In figure 1, the value of \( b/v^3 \) is plotted; \( b \) seems to be growing faster than cubically.

![Figure 1: \( b/v^3 \) vs. \( v \)](image)

3. Lower bound on \( b \)

The computations of [1] show that there is a single basic minimal 1-(7,3) design of length 21; figure 2 shows the columns, with their multiplicities. This has been arranged so that a pattern may be seen, which may be generalized (some designs of length 10 were also examined).

![Figure 2. 1-(7,3) design of length 21](image)

**Theorem 4.** If \( v = 3t + 1 \) where \( t \geq 2 \) then there is a 1-(v,3) design with \( b = (2t - 1)v \).
Proof. Let $B$ denote the $v \times v$ matrix of the basic columns of $M$. Row 0 of $B$ consists of $2t + 1$ 1’s, followed by 0’s. Column $2t - 1 + j$ for $0 \leq j < t$ has 1’s in rows $3j + 1, 3j + 2, 3j + 3$. Define $m_1 = 6t - 5$, $m_2 = 4t - 3$, $m_3 = 2t - 1$; let $I$ denote an identity matrix, $J$ an all 1’s matrix, and $K$ the $2t - 4 \times t - 2$ 0-1 matrix which has 1’s in rows $2j$ and $2j + 1$ in column $j$. Rows 1 to $3t$ and columns 0 to $2t + 1$ are given by the following block matrix:

\[
\begin{array}{cccc}
1 & t & t - 2 & 2 \\
\hline
\text{t} & 0 & I & 0 & 0 \\
2t - 4 & 0 & 0 & K & 0 \\
1 & 0 & 0 & 0 & J \\
2 & J & 0 & 0 & I \\
1 & 0 & J & 0 & 0 \\
m_3 & 2 & 2 & 1
\end{array}
\]

The left column gives the block heights, the top row gives the block widths, and the bottom row the multiplicities. Columns $2t + 1$ through $3t - 1$ have multiplicity $m_1$, and column $3t$ has multiplicity $m_2$. One readily verifies that the rows of the design have $6t - 3$ 1’s.

4. Bounding $x_{\text{max}}$

Solutions with $x_{[0,...,k-1]} = k/v$ are clearly of little interest. This suggests modifying the LP by adding constraints $x_C + y_C = x_{\text{max}}$, $y_C \geq 0$. Basic feasible solutions of $M^+_D x = e_0$, with integer value having a greatest common divisor of 1, are minimal; this is a fact of LP theory, tacitly assumed in [1]. With slack variables added as just indicated, the restriction of a basic solution to the $x_C$ may no longer be basic in the original LP.

The cost of the slack variables must be specified. In ordinary use this is 0 (see section 3-2 of [2]). For this paper, this value is adopted.

By results of [1], for a minimal solution which is not basic, $b \leq 588$. The modified LP was run with values of $x_{\text{max}} = n/d$, where $2 \leq d \leq 84$ or $d \leq 588$ and $d \mod 7 = 0$, $1/7 \leq n/d \leq 3/7$, and $\gcd(n, d) = 1$ (there are 4333 such $n/d$).
Each LP has a solution where the cost is $x_{\text{max}}$; further $b$ equals $d$ if $d \mod 7 = 0$, else $7d$. Using the 33395 basic minimal solutions (see [1]), it may be determined that for each modified LP, the optimum solution found by the simplex method is an integral linear combination of basic minimal solutions. In the cases where $x_{\text{max}}$ is the cost of a basic minimal solution, the optimal solution is basic minimal. These facts suggest that for $v = 7$, $k = 3$, minimal solutions are basic minimal; further remarks are omitted here.

References

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