A LOWER BOUND ON THE LENGTH OF BASIC MINIMAL 1-(3t+1,3) DESIGNS

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Abstract: In a previous paper the author found some minimal 1-(v,3) designs with large b, by exhaustive search. Here some further such are found, by more ad-hoc methods. A construction is given for $v = 3t + 1$ for $t \geq 2$, where b grows quadratically with v.

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1. Introduction

Suppose v and k are integers with $1 \leq k \leq v$. A 1-design with parameters v and k is a v row by b column matrix for some b, with elements either 0 or 1, such that each column contains k 1's, and each row contains r 1's for some r. Such a matrix is exactly the incidence matrix of a "biregular bipartite graph", with v vertices of degree r in one class, and b vertices of degree k in the other class. The notation "1-(v,k) design will be used to denote such designs.

There is a well-known system of linear equations associated with 1-designs. Various facts of interest about this system are proved in [1]. The following further fact is of interest, both for the theory of 1-designs and for computations.

Theorem 1. *Suppose D is a basic minimal 1-design, and S is the set of its columns. Then D is determined by S.*

Proof. Suppose first that S is nondegenerate, i.e., $|S| = v$ where there are v rows. Let M be the matrix $[C_1, \ldots, C_v]$ where $S = \{C_1, \ldots, C_v\}$. Let x be the column vector of indeterminates $x_{C_1}, \ldots, x_{C_v}$. The system $Mx = J_{v\times 1}$ (see [1] for notation) has a unique solution x. D is obtained when x is converted to an integer vector with the integers having no common divisor. In general, suppose T_1 and T_2 are extensions of S to bases. It is readily seen that in the primal simplex tableau there is a sequence of level 0 pivots from T_1 to T_2. Further, the solutions for T_1 and T_2 have the same restrictions to $\{x_C : C \in S\}$. See [5] for terminology. □

2. Long designs

Theorem 6 and 7 of [1] give bounds on the length b of minimal and basic minimal 1-(v,k) designs. It is of interest to both design theory and to linear programming theory to obtain improvements to these bounds, and to obtain lower bounds. As seen in [1], these questions are of interest even for $k = 2$.

For $5 \le v \le 8$ the exact bound for basic minimal 1-designs was determined in [1] by exhaustive search using vertex enumeration. For $k = 3$ these values are b=5, b=21, and b=48 respectively. (There is an error in Table 1 of [1]; the number of non-isomorphic basic minimal 1-(6,3) designs is 3, not 4).

In this paper, a lower bounds for basic minimal solutions with $k = 3$ will be given. Before giving this, a method for obtaining examples and lower bounds for small v will be given, which uses random search rather than exhaustive search.

Using notation as in [1], let M be the matrix with v rows labeled $0, \ldots, v-1$ and $\binom{v}{k}$ columns, the k element subsets of $\{0, \ldots, v-1\}$. Let M_H^+ be the matrix derived from M by subtracting row 0 from the other rows and replacing it by a row of all 1's. A vector x is a solution to $Mx = J_{v\times 1}$ iff $y = (v/k)x$ is a solution to $M_H^+ y = e_0$ where e_s denotes the (column) unit vector with 1 in row s. It follows that the null spaces of M and M_H^+ are the same, and so the sets of columns which are bases is the same (labeling a column of M_H^+ with the column of M from which it is derived).

Solutions to the LP $M_H^+ x = e_0$, $x_C \geq 0$ for all C will be considered. This LP is in standard form, and is readily solved using the primal simplex method, as described in [5] for example. The two-stage method for finding an initial feasible basis may be avoided, if a feasible basis is known. The columns of it may be transformed into distinct unit vectors by an arbitrary sequence of pivots on the columns of the basis.

For $\gcd(v,k) = 1$ define the cyclic design to be that where column i is $\{0 + i \bmod v, \ldots, k-1+i \bmod v\}$.

Theorem 2. *For* $\gcd(v,k) = 1$ *the cyclic design is basic.*

Proof. Since $\gcd(v,k) = 1$, $1 + x + \cdots + x^{k-1}$ and $x^v - 1$ are relatively prime. The theorem follows by circulant matrix theory (see [3]). □

Theorem 3. *Suppose* $\gcd(v,k) = 1$ *and* $v = qk + s$ *where* $q \geq 2$ *and* $0 < s < k$. *Let* x *be the vector where* $x_C = k/v$ *if* C *has 1's in rows* kt *to* $kt + k - 1$ *for some* t *with* $0 \leq t \leq q-2$*;* $x_C = 1/v$ *if* C *is one of the columns of the cyclic design in rows* $k(t-1)$ *to* $v-1$*; and* $x_C = 0$ *otherwise. Then* x *is a solution to* $M_H^+ = e_0$ *which maximizes* $x_{\{0,\ldots,k-1\}}$.

Proof. Let B be the matrix where for $j < v-k$ column j has 1's in rows $j, \ldots, j+k-1$; and in the remaining columns there is a copy of the cyclic design in the lower right. B is block upper triangular, where the blocks along the diagonal are invertible (using theorem 2), so B is basic. Direct computation shows that $Bx = (k/v)J_{v\times 1}$. Direct computation also shows that the top row of $(B_H^+)^{-1}$ has k/v in column 0 and $-1/v$ in the other columns. Let c be the (row) cost vector, with -1 in column 0 and 0's elsewhere. Let c_B be the similarly defined vector of length v. As noted in section 2.6 of [5], the relative cost vector of the simplex tableau at the basis B equals $c - c_B(B_H^+)^{-1}M_H^+$. Direct computation shows that this is 1 in column C if $0 \in C$, except for column 0, which is 0; and 0 in the remaining columns. □

The maximum value of x_C for the cyclic design is $1/v$. Starting from this basis, the simplex algorithm may be run with random pivots. The basic feasible solution with the largest value of b can be monitored. For $k = 3$, this was done for $7 \leq v \leq 20$ with

$\gcd(v,3) = 1$. For a given v, 100,000 repetitions were performed. In figure 1, the value of b/v^3 is plotted; b seems to be growing faster than cubically.

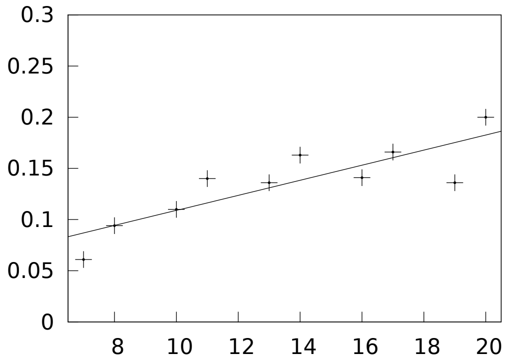

Figure 1: b/v^3 vs. v

3. Lower bound on b

The computations of [1] show that there is a single basic minimal 1-(7,3) design of length 21; figure 2 shows the columns, with their multiplicities. This has been arranged so that a pattern may be seen, which may be generalized (some designs of length 10 were also examined).

x	x	x	x	x		
	x				x	
		x			x	
			x	x	x	
x			x			x
x				x		x
	x	x				x
3	2	2	1	1	7	5

Figure 2. 1-(7,3) design of length 21

Theorem 4. *If $v = 3t + 1$ where $t \geq 2$ then there is a 1-(v,3) design with $b = (2t-1)v$.*

Proof. Let B denote the $v \times v$ matrix of the basic columns of M. Row 0 of B consists of $2t+1$ 1's, followed by t 0's. Column $2t-1+j$ for $0 \le j < t$ has 1's in rows $3j+1, 3j+2, 3j+3$. Define $m_1 = 6t-5$, $m_2 = 4t-3$, $m_3 = 2t-1$; let I denote an identity matrix, J an all 1's matrix, and K the $2t-4 \times t-2$ 0-1 matrix which has 1's in rows $2j$ and $2j+1$ in column j. Rows 1 to $3t$ and columns 0 to $2t+1$ are given by the following block matrix:

	1	t	$t-2$	2
t	0	I	0	0
$2t-4$	0	0	K	0
1	0	0	0	J
2	J	0	0	I
1	0	J	0	0
	m_3	2	2	1

The left column gives the block heights, the top row gives the block widths, and the bottom row the multiplicities. Columns $2t+1$ through $3t-1$ have multiplicity m_1, and column $3t$ has multiplicity m_2. One readily verifies that the rows of the design have $6t-3$ 1's. ☐

4. Bounding $x_{\max}$

Solutions with $x_{\{0,\ldots,k-1\}} = k/v$ are clearly of little interest. This suggests modifying the LP by adding constraints $x_C + y_C = x_{\max}$, $y_C \ge 0$. Basic feasible solutions of $M_H^+ x = e_0$, with integer value having a greatest common divisor of 1, are minimal; this is a fact of LP theory, tacitly assumed in [1]. With slack variables added as just indicated, the restriction of a basic solution to the x_C may no longer be basic in the original LP.

The cost of the slack variables must be specified. In ordinary use this is 0 (see section 3-2 of [2]). For this paper, this value is adopted.

By results of [1], for a minimal solution which is not basic, $b \le 588$. The modified LP was run with values of $x_{\max} = n/d$, where $2 \le d \le 84$ or $d \le 588$ and $d \bmod 7 = 0$, $1/7 \le n/d \le 3/7$, and $\gcd(n,d) = 1$ (there are 4333 such n/d).

Each LP has a solution where the cost is $x_{\max}$; further b equals d if $d \bmod 7 = 0$, else $7d$. Using the 33395 basic minimal solutions (see [1]), it may be determined that for each modified LP, the optimum solution found by the simplex method is an integral linear combination of basic minimal solutions. In the cases where $x_{\max}$ is the cost of a basic minimal solution, the optimal solution is basic minimal. These facts suggest that for $v = 7$, $k = 3$, minimal solutions are basic minimal; further remarks are omitted here.

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