Some Mars Trajectory Optimizations

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Abstract. In an earlier paper an integrated space program was outlined, which includes the capability for manned Mars missions. The time for the manned flight of a "split" mission with aerobraking was estimated at around one year. Here some trajectory optimizations are given which demonstrate that lower flight times are realistic, with chemical propulsion.

1. Introduction. This paper presents some basic Mars trajectory optimizations. These provide some substantiation of the feasability of suggestions made in [Do02], indeed of improved suggestions. Estimates were given in [Do02], which relied on earlier, unpublished, optimizations. New optimizations are given here.

The universal anomaly provides a uniform method for solving astrodynamics problems. For example, in [Do88] a Newtonian iteration for determining the orbital parameters from three Earth observations is given. The universal anomaly as treated in [Do88] is reviewed in appendix 1 (including a correction to an error). The next section presents some futher identities, which were useful in the computation for later sections. Appendix 2 contains a summary of formulas regarding the partial derivatives which are useful in orbit reduction.

2. Universal anomaly. The universal anomaly can be made dimensionless by writing ψ as $\sqrt{p/\mu}\Psi$. Also (using $\alpha = (\mu/p)(e^2-1)$), $s_j = (p/\mu)^{j/2}S_j$ where

$$S_j(\Psi) = \sum_i \frac{(e^2-1)^i\Psi^{2i+j}}{(2i+j)!}.$$

Other quantities can be made dimensionless. Letting $\rho = r/p$, the orbit equation becomes $\rho = 1/(1+e\cos\nu)$; and the dimensionless version of the energy equation is $\mathcal{E}/(\mu/p) = (v/\sqrt{\mu/p})^2/2 - (1/\rho) = (e^2-1)/2$.

With these definitions, provided ν and Ψ are nonnegative,

$$d\Psi = \frac{d\rho}{\sqrt{-1+2\rho-(1-e^2)\rho^2}}.$$

Integrating, choosing the constant of integration so that $\Psi = 0$ when $\rho = 1/(1+e)$ or $\nu = 0$, and using the normalized orbit equation,

$$\Psi = \begin{cases} \dfrac{1}{\sqrt{1-e^2}}\cos^{-1}\Big(\dfrac{1-(1-e^2)\rho}{e}\Big) = \dfrac{1}{\sqrt{1-e^2}}\cos^{-1}\Big(\dfrac{e+\cos\nu}{1+e\cos\nu}\Big), & e<1 \\ \sqrt{2\rho-1} = \sqrt{\dfrac{1-\cos\nu}{1+\cos\nu}}, & e=1 \\ \dfrac{1}{\sqrt{e^2-1}}\cosh^{-1}\Big(\dfrac{1-(1-e^2)\rho}{e}\Big) = \dfrac{1}{\sqrt{e^2-1}}\cosh^{-1}\Big(\dfrac{e+\cos\nu}{1+e\cos\nu}\Big), & e>1 \end{cases}$$

The correct period and sign must be determined; one convention for the latter is that Ψ and ν are negative for inbound points. In the case $e = 0$ the expression in ρ is undefined, and $\Psi = \nu$.

Also,

$$S_0 = \frac{1-(1-e^2)\rho}{e} = \frac{e+\cos\nu}{1+e\cos\nu} = \begin{cases} \cos(\sqrt{1-e^2}\Psi), & e<1 \\ 1, & e=1 \\ \cosh(\sqrt{e^2-1}\Psi), & e>1 \end{cases}$$

and

$$S_1 = \frac{\sqrt{-1+2\rho-(1-e^2)\rho^2}}{e} = \frac{\sin\nu}{1+e\cos\nu} = \begin{cases} \frac{1}{\sqrt{1-e^2}}\sin(\sqrt{1-e^2}\Psi), & e<1 \\ \Psi, & e=1 \\ \frac{1}{\sqrt{e^2-1}}\sinh(\sqrt{e^2-1}\Psi), & e>1 \end{cases}$$

For $i \geq 2$ S_i can be computed using $S_i = \Psi^i/i! + (e^2-1)S_{i+2}$.

Choosing t_0 as a time of periapsis,

$$\frac{(t-t_0)}{\sqrt{p^3/\mu}} = S_1/(1+e) + S_3 = \begin{cases} (1-e^2)^{-3/2}(\sqrt{1-e^2}\Psi - e\sin(\sqrt{1-e^2}\Psi)), & e<1 \\ \Psi/2+\Psi^3/6, & e=1 \\ (e^2-1)^{-3/2}(e\sinh(\sqrt{e^2-1}\Psi) - \sqrt{e^2-1}\Psi), & e>1 \end{cases}$$

Comparing to equation (2.12) of [PR93], in the case of elliptic orbits $\sqrt{1-e^2}\Psi = E$ where E is the eccentric anomaly. The equation for t becomes $t = t_0 + \sqrt{a^3/\mu}(E - e\sin E)$, which is a standard form for the Kepler equation. Determining Ψ from t (or its dimensionless value $(t-t_0)/\sqrt{p^3/\mu}$) is a variant of the Kepler problem and requires iteration; in contrast $\Psi(\rho)$ and $\Psi(\nu)$ were given above using elementary functions.

Although not used in this paper, a power series for Ψ is readily obtained, namely

$$\Psi = \int \frac{d\rho}{\sqrt{-1+2\rho-(1-e^2)\rho}} = \sum_{i\geq 0} \frac{(2i)!}{2^{2i}(i!)^2}\left(\frac{1}{2^{i+1}}\sum_{j=0}^{2i}\binom{2i}{j}\frac{(2\rho-1)^{i-j+1/2}}{i-j+1/2}\right)(1-e^2)^i.$$

This is useful when $e \approx 1$ and $(1-e)\rho \approx 0$.

3. Optimum trajectories. In a circular orbit at radius r_c, a new orbit is achieved by a boost velocity vector, which may be specified by its magnitude v_b and angle ϕ from the tangent to the circular orbit. Letting v_c denote $\sqrt{\mu/r_c}$, the velocity in the circular orbit, the initial position and velocity in the new orbit are $\mathbf{r} = \langle r_c, 0\rangle$ and $\mathbf{v} = \langle v_b\sin\phi, v_c + v_b\cos\phi\rangle$ (so that positive ϕ yields an outward boost). The orbital parameters are readily computed from v_b and ϕ; indeed $v^2 = v_b^2 + 2v_bv_c\cos\phi + v_c^2$, $h = r_c(v_c + v_b\cos\phi)$, $p = h^2/\mu$, and $e = \sqrt{1+\alpha p/\mu}$.

If it is assumed that aerobraking is done at the target planet (Mars outbound, Earth inbound), there are two boosts (at Earth outbound, at Mars inbound). With both boost magnitudes fixed, and also the time of stay at Mars, the flight angles may to be chosen to minimize the total trip time, subject to the constraint that the return flight arrive at Earth in its new position. The constraint may be written as $\nu_e + t_s(2\pi/y_m) + \nu_m = (t_e + t_s + t_m)(2\pi/y_e)$ where t_s is the time of stay, y_e (y_m)

is the period of revolution of Earth (Mars), and t and ν are the changes of these orbit variables during the flight. The quantities $\nu_e - t_e(2\pi/y_e)$, $t_s(2\pi/y_m - 2\pi/y_e)$, and $\nu_m - t_m(2\pi/y_e)$ are the angular leads (negative leads being lags) of the spacecraft versus Earth for the three phases of the trip.

Some hopefully representative values have been selected for various parameters. (Unless otherwise indicated, units are SI). The lag due to the difference of the Earth and Mars revolution rates was set at .2 radians, which corresponds to a time of stay of 24.8 days, in the "short stay" class of mission.

To determine a reasonable domain of boost magnitudes, the ratio of fuel mass to payload mass required for the boost should be estimated, for which estimates of the exhaust velocity and tank mass are required. The value 4460 is used for the exhaust velocity, being often stated as the value for the SSME (space shuttle main engine) in a vacuum. Higher velocities are considered in the literature, especially if flourine oxidizer is used. A value of 5% of the fuel mass is used for the tank mass. This may be somewhat low, especially since the tank has to withstand being placed in orbit. More careful estimates of parameters such as this are outside the scope of this paper.

The formula for determining the velocity delta v_δ from the exhaust velocity v_E, the fuel mass m_F in units of payload mass, and the tank weight factor τ is $v_\delta/v_E = \ln(((1+\tau)m_F+1)/(\tau m_F+1))$. The velocity delta does not equal the interplanetary boost velocity, since the spacecraft starts with a circular velocity v_c around a planet. The velocities are related by the formula $(v_\delta+v_c)^2 = 2v_c^2+v_b^2$, the left side being the periapsis velocity squared and v_b being the asymptotic velocity, in hyperbolic orbit. The values 7700 and 3500 are used for v_c at Earth and Mars.

The following table shows how m_F varies with v_b at Mars, for values from 8000 to 9500.

v_b	8000	8500	9000	9500
v_δ	5907	6336	6771	7212
m_F	3.20	3.72	4.34	5.06

The following table shows how m_F varies with v_b at Earth, for values from 13000 to 17000; the fuel mass m_{F2} for two stages is also given. It appears that two stages should be used for the Earth burn.

v_b	13000	14000	15000	16000	17000
v_δ	9258	10036	10836	11654	12489
m_F	10.70	14.76	21.47	34.35	67.85
m_{F2}	8.24	10.31	12.96	16.40	20.89

Estimating the payload mass will require substantial research, outside the scope of this paper, but a rough estimate of 20000 Kg for a two man vehicle with life support and aerobraking capsule can be used to get some idea of the initial mass in LEO. Values as high as 20 for m_{F2} might be within the realm of possibility; however the returns are diminishing, and as will be seen the Mars entry velocity limits the Earth boost. For this section the above domains were used for the boosts;

for the lowest two values at Mars, the Earth boost of 13000 is excluded. The question of getting the mass to LEO is not considered, except to note that the fuel must be sent up in batches, so the boiloff problem must be solved; and in this case modest sized batches can be used (40 tons for example).

It is worth remarking on how the above tables were calculated. In the case of a single stage, m_F can be determined from v_δ by inverting the above formula, giving $m_F = (e^{v_\delta/v_E} - 1)/(1 + \tau - \tau e^{v_\delta/v_E})$. For two stages, an optimization must be done over the split for a given m_{F2}, and the value of this giving the desired boost determined. This was done using two C functions, `x_y` and `x_min`, each with a function argument.

These routines were used to accomplish the foregoing and any other cases of these tasks throughout the calculations. For example, the Kepler problem would be readily solved using `x_y`. These routines use only function evaluations, and do not bother with derivatives. This reduces the programming effort, and handles cases where derivatives are difficult or impossible to compute. Parabolic approximation is used to accelerate convergence.

The strategy for obtaining the optimum flight angles is to vary ϕ_e at Earth, compute ν_e and t_e for the outbound flight, determine the value ϕ_m at Mars which gives the required value of the Mars lag, and compute the time t_m for this. To avoid searching to compute ϕ_m from the lag, a sample is taken over an appropriate domain of ϕ_m, and a cubic Bessel interpolation [deB78] done.

To fully automate the process, flight angle domains for which the calculation succeeds must be determined. For Mars, the maximum is the value between π and 2π at which the periapsis distance equals the radius of Earth's orbit; and the minimum is the value between π and the maximum which minimizes the time. For Earth, first determine the value between $-\pi$ and 0 at which the apoapsis distance equals the radius of Earth's orbit (if the boost yields hyperbolic orbits the domain must first be restricted to elliptic orbits). Then determine the value between this and 0 where the lead is maximized. Finally, intersect this domain with that giving the required range of Mars lags.

Figure 1 is a graph of the optimum time in days as a function of the Earth boost magnitude (13000 to 17000), for each Mars boost magnitude (8000 to 9500). It could be argued that the point of diminishing returns would seem to be below the highest boosts, with a Mars boost of 8000 and an Earth boost of 15000 to 16000 being a region of interest; some specific pairs are discussed in more detail below.

Lower boosts are advantageous not only in requiring a smaller booster, but a smaller amount of decelaration at the target planet also. The formula relating the velocity change v_δ between an orbit with parameters p and e and a circular orbit at radius r is

$$v_\delta^2 = \mu\Big(\frac{3}{r} - \frac{(1-e^2)}{p} - \frac{2\sqrt{p}}{\sqrt{r^3}}\Big).$$

Treating v_b as v_∞ at the target planet, the entry velocity is v_{per} where as above $v_{per}^2 = 2v_c^2 + v_\infty^2$ (the actual atmospheric entry velocity depends on the approach and will be somewhat different).

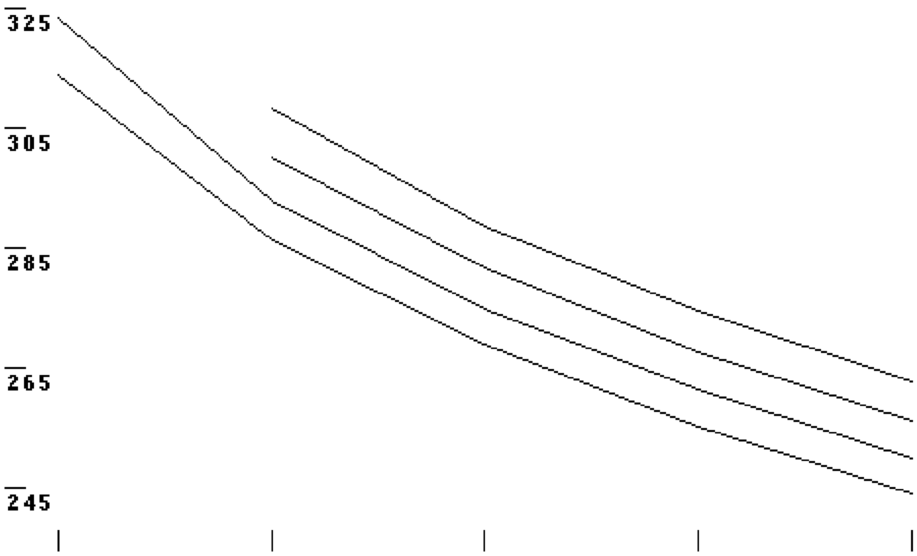

Figure 1. Optimum times.

The following table shows various values for low, intermediate, and high boosts. O (I) denotes the value for the outbound (inbound) flight; v_b is the interplanetary boost, m_F is the fuel mass, v_e is the entry velocity at the target planet, t is the time in days, with t_T being the total time (including the stay time). Figure 2 shows the trajectories for $v_{bO} = 15500$ and $v_{bI} = 8000$.

v_{bO}	m_{F2O}	v_{eO}	t_O	v_{bI}	m_{FI}	v_{eI}	t_I	t_T
14000	10.31	11759	186.5	8000	3.20	14799	96.7	308.1
15500	14.57	14624	162.4	8000	3.20	15638	93.9	281.0
17000	20.90	18223	135.5	9500	5.06	15954	83.7	244.1

4. Aerobraking. Entry velocities of 15 Km/sec are higher than in previous missions, both at Earth (10 for Apollo, 12.5 for Startdust) and Mars (7.5 for Pathfinder); but considerably lower than Galileo (47.4). Although no attempt at a detailed justification will be made here, the literature on aerobraking (for example [Cr80], [Ha88], [SI99]) suggests that the aerobraking required is an application of known methods. At Mars the deceleration is from ~ 14500 to ~ 3500, which can be accomplished at 80 for 140 sec; this is within published limits for human endurance.

The entry trajectory is such that the entry capsule ends up in LMO (immediate descent to the surface should be considered, but overall logistics seem to favor orbital entry of a small capsule). Although this remains to be demonstrated, it seems likely that a suitable trajectory can be achieved using active control with rockets. Multisensory input can be used, including optical, radar altimeter, inertial, and telemetry (such as the density profile). Attitude control also uses the rockets (in particular spin stabilization is not required). The trajectory can accomplish plane change, although the simulation here is in two dimensions.

Planetary entry can make use of constant lift to drag, which yields a downward force ap-

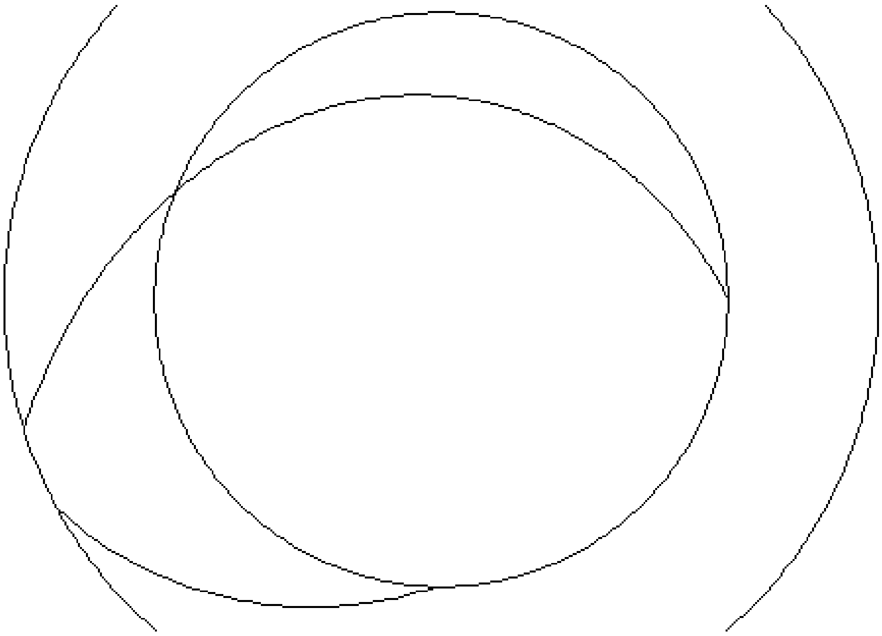

Figure 2. Optimum flight.

proximately balancing the centrifugal force, to keep the craft in the atmosphere longer for greater velocity reduction. Simulations show that for Mars entry, the lift to drag should be around .6, and that the final burn mass ratio can be considerably reduced by variation. The approach altitude should be conservative due to the sensitivity to it. A schematic sketch of the entry capsule is shown in Figure 3. It has an ablating front surface, a couch for each of 2 astronauts, and a diameter perhaps somewhat under 5 meters. The nose is shown bent, giving nonzero neutral lift to drag.

Rockets on the rear are used for the final burn (requiring a 180 degree pitch), lift to drag variation, and attitude control (additional rear mounted attitude control rockets are not shown). Rockets on the side facilitate adjusting the approach trajectory, and provide extra control and safety margins (although an additional design problem).

The astronauts are in an inner chamber; only this is pressurized. The cross section is not necessarily circular, being perhaps closer to a rounded square; this is true of the inner chamber also. Couches for the astronauts are shown. Also shown is an attachment ring; this allows closing a hatch and separating, and later docking with the depot and opening the hatch.

After achieving orbit the inner chamber separates from the capsule; the astronauts do not exit from it until it has docked with the depot. Separation can take place immediately upon achieving orbit, solving the residual heat problem. The astronauts wait for a "sled", which has been dispatched from the equipment depot to the predicted entry location. The inner chamber makes a connection to it and is ferried to the depot, where it docks with the depot. The sled can be equipped to handle a range of final orbits.

As has been seen the questions of the capsule shape; the size, placement, and usage of the

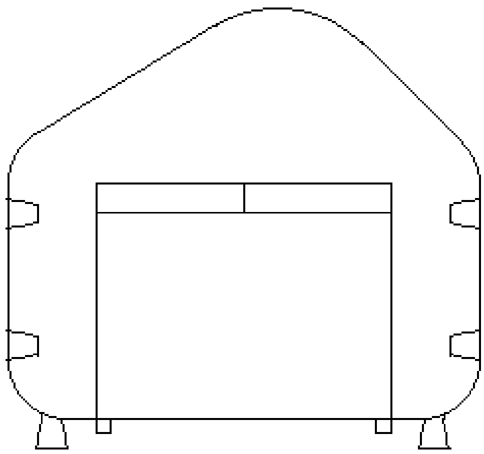

Figure 3. Entry capsule.

rockets; and the control algorithm and its tolerance must be given satisfactory answers to be certain that this method is feasible and reliable. The simulation done here identifies some considerations, in particular giving example trajectories.

Parameters concerning Mars used for the simulation are as follows. The mass of Mars is 6.42×10^{23}; μ denotes the gravitational parameter GM where $G = 6.67259 \times 10^{-11}$. The radius R_M of Mars is 3.39×10^6. The boundary of Mars' atmosphere H_B is 240 Km (simulation of drag begins and ends at 240 Km). The density ρ of the atmosphere as a function of the altitude has a piecewise linear log, with some sample values taken from the literature as the breakpoints; at 10 Km it is 7×10^{-3}, and at 240 Km 10^{-12}.

The aerobrake surface area S is taken as 12.5. The drag coefficient C_D is taken as 2 above 120 Km, decreasing linearly to 1.7 at 75 Km, and 1.7 below 75 Km (this is from Pathfinder data; a more realistic value is impossible to estimate without extensive design and simulation). The formula $C_D(r)S(\rho(r)v^2/2)(\mathbf{v}/v)$ is used for the drag. A downward radial acceleration is added, which is equal to a parameter λ times the drag; this value is a constant plus a variation, which is taken as a function of the velocity magnitude.

The simulation interval is set at 1/8-sec. The initial conditions (at 240 Km altitude) are determined by the orbital parameter p of the approach orbit. The asymptotic inbound velocity v_∞ is taken as 13800. When the capsule again reaches 240 Km, it moves in orbit up to 300 Km, and the circularization impulse is determined.

The initial values of $\mathbf{r}$ and $\mathbf{v}$ are determined as in [PC93], section 3.3, with some simplifications worth noting. Rather than p, the parameter is the altitude H_P at periapsis in free flight; then

$r_{per} = R_M + H_P$, $r = R_M + H_B$, $e = 1 + r_{per}v_\infty^2/\mu$, $p = r_{per}(e+1)$, $\cos\nu = (p/r - 1)/e$, $\sin\nu = -\sqrt{1-\cos^2\nu}$, $v = \sqrt{\mu(2/r - (1-e^2)/p}$, $v_\perp = \sqrt{\mu p}/r$, and $v_\parallel = -\sqrt{v^2 - v_\perp^2}$. From these $\mathbf{r}$ and $\mathbf{v}$ are readily determined, using $\cos\nu$ and $\sin(\nu)$ to "rotate" the v components.

For comparison, the impulse required when $H_P = 300$ and there is no aerobraking is 11210, and `exp`(11210/4460)=12.35. With $H_P = 111$ Km and $\lambda = .6$ the mass ratio is 5.38; the capsule does not remain low enough in the atmosphere to achieve greater amounts of braking. To solve this problem a variation is applied to λ.

The variation is a function of the velocity, normalized to a value between 0 and 1 from the final orbit velocity to the entry velocity. The function is a clamped uniform B-spline of degree 3. An exhaustive search with dimension 7 found three local minima, with coefficient vectors

$.01, .05, -.07, -.06, -.08, -.08, .1$ (mass ratio 1.435)

$.07, -.05, -.09, -.08, .03, -.1, .1$ (mass ratio 1.482)

$.09, -.09, -.08, -.07, -.01, -.09, .1$ (mass ratio 1.452)

Searching some more around the third found the coefficient vector

$.09, -.09, -.081, -.065, -.017, -.073, .13$ (mass ratio 1.161).

The last trajectory is shown in figure 4; The maximum acceleration is 87.8 m/sec^2 at 88 sec, the minimum altitude is 103.2 Km at 113 sec, the time above 1g is 357 sec, the time to atmospheric exit is 886 sec, and the time from exit to final orbit (not shown) is 81 sec.

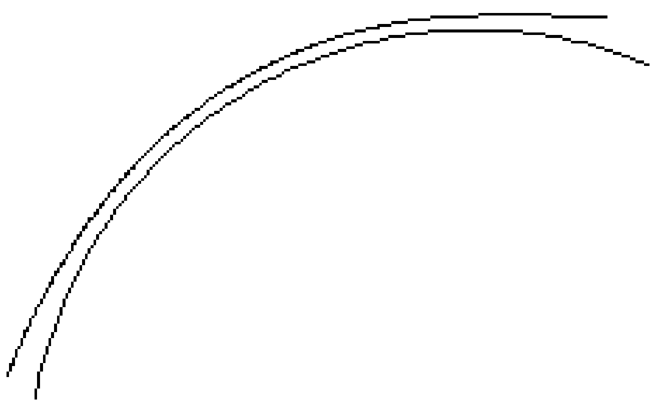

Figure 4. Entry trajectory.

Further research on these trajectories is required, in particular taking in to account the fuel required for the control; and determining the trajectory and the probability of hitting the surface as a function of H_P. The fuel mass required seems to be sufficiently low that oxy-hydrogen can be used (which was assumed above); non-cryogenic alternatives might also be considered.

To conclude this section an operational plan for the outbound flight is outlined. Figure 5 shows the initial configuration in LEO, approximately to scale with a total length around 45 meters. Figure 6 shows a detail of the interplanetary vehicle in retracted position. The articulated arms are double beams with cross members. After achieving interplanetary orbit, the arms are extended and the structure set in rotation so that the astronauts travel in 1 earth gravity. Power is provided by a "despun" pole with redundant solar thermal heat engine power plants; this contains the communications antenna also.

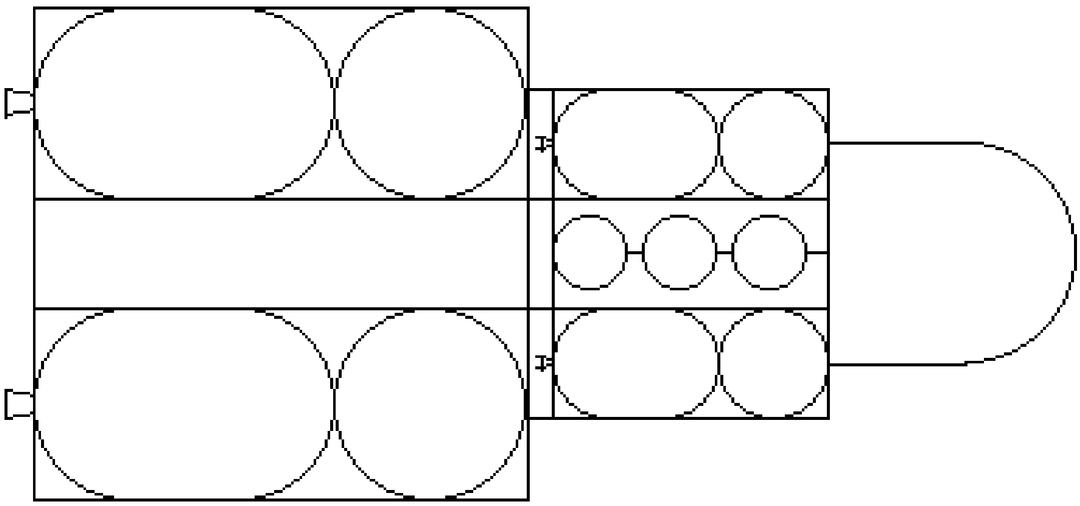

Figure 5. Outbound crew vechicle.

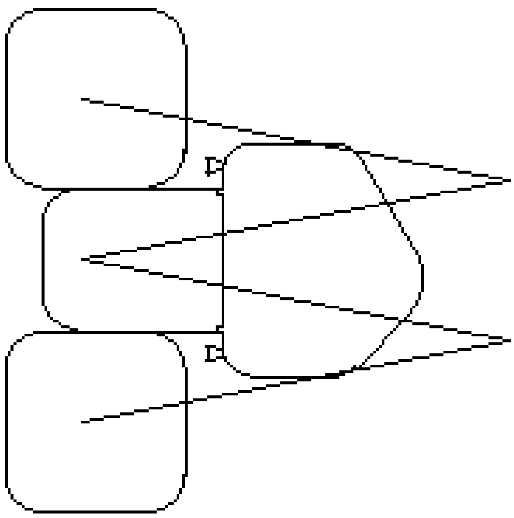

Figure 6. Detail of figure 5.

Some time prior to Mars entry, the rotation is stopped and the transit capsules retracted to the center, where they form a pressure seal with a central chamber, which also has a pressure seal with the inner chamber of the entry capsule. Finally the astronauts enter the inner chamber, and the entry capsule separates.

The estimate of 20000 Kg for the mass of the transit vehicle might be broken down as follows: 2000 for the entry capsule weight, 6500 each for the transit capsules, and 5000 for the rest. To reduce failure probability redundancy can be applied where possible.

A complete list of the crew compartments for a mission of this type is as follows, starting in LEO and ending in an LEO.

- An interplanetary transit compartment for each of the two crew members. At LEO boost these are in retracted position; they are extended after boost to give 1G for interplanetary transit. They are retracted upon arrival in Mars region, and form hatchways with a crawlway.
- The crawlway in turn has a hatchway to the Mars entry compartment; this must separate from the crawlway, connect to the "sled", and dock with the depot.
- The depot has a main compartment. This has 4 hatchways, for the Mars entry compartment, the descent / ascent compartment, and an interplanetary return transit compartment for each astronaut.
- The ascent / descent compartment has a second hatchway, to a tube which descends to the Mars habitation compartment. The tube seems necessary, since the ascent compartment must be high on the landing platform, and the habitation compartment low.
- The habitation compartment has a second hatchway, to the rover.
- The interplanetary return transit compartments are initially docked with the main depot compartment, and remain so until the return sequence begins; the return vehicle is mechanically coupled. Deployment into return interplanetary rotation is accomplished from this configuration (the entry compartment and ascent compartment can remain attached to the main depot compartment; separation consists of mechanically uncoupling from the main compartment).
- Retraction upon arriving at the region of Earth establishes hatchways to the Earth injection vehicle; the inbound extension / retraction thus starts at one compartment and ends at another.

At Earth direct entry to the ground and oceanic recovery is possibly more advantageous than entry into orbit.

5. Equipment mission. The rise in m_F from 3.2 to 5 when the inbound boost rises from 8000 to 9500 does not represent this large a rise in the mass at LEO, due to the extra payload mass of the other equipment. Example figures of 20000 Kg for the return vehicle payload and 25000 Kg for the other equipment will be used. The other equipment consists of the sled, an orbiter for one of the astronauts, and a lander for the second astronaut. The lander includes a ROV which is operated by the orbiting astronaut, for longer range exploration than the landing astronaut, who has a short range rover. Both rovers gather samples for return.

The power plant might be a variant of a standardized reflector and heat engine variety, except the reflector is deployed. To simplify this, and to have both the primary reflector and the heat engine on the ground, a secondary mirror might be used. The primary mirror might then use ruled surfaces, simplifying deployment. Batteries are charged by the heat engine, including those for the manned rover.

The figure of 25000 for the equipment mass is perhaps optimistic; values as high as 35000 are well within the ability of the program. Rough estimates of the budget of the 25000 Kg equipment mass are 4000 for the orbital equipment, with 20% of the remaining 21000 for aeroshell, parachute, and braking rockets, placing slightly under 17000 Kg on the surface. This size of surface cargo packet was considered in the Mars reference mission. Scaling from Pathfinder, the parachute diameter would be on the order of 80 meters. The braking rocket thrust is around 600000 N.

In [Do02] the proposed method for the equipment flight involved both solar thermal and nuclear ion propulsion, with propulsive deceleration at Mars. Here a spiral flight involving only solar thermal ion propulsion, and aerobraking at Mars, is considerded. An additional 15000 Kg is used as an estimate of the mass needed to place the equipment in LMO using aerobraking. A rough estimate of the shield dimensions is 15 by 30 meters; this can be put into LEO in 2 sections, using the "large object vehicle" described in [Do02].

The mass m_M to be sent to Mars is $20000 + 65000 + 25000 + 15000 = 125000$ Kg (return vehicle, fuel, Mars equipment, entry surface). The mass placed in LEO equals this, plus the ion power plant mass m_P, plus the ion fuel mass m_F. The power plant is assumed to produce 25 W per Kg. The split between m_P and m_F can be optimized, for a fixed sum.

The power plant detaches once interplanetary orbit is achieved, and returns to LEO. Interplanetary orbit is a hyperbolic orbit around Earth, with asymptotic velocity that required for Hohmann transfer to Mars (2945); the LEO orbit is assumed to have altitude 500 Km. The exit direction from Earth orbit will be ignored; it can be achieved by stopping and / or thrust vectoring the ion thrusters in the final stages of the spiral.

As can be found in many freshman physics texts, the rocket equation in one dimension is $m\dot{v} = \dot{m}v_E$ where $m = m_0 - \dot{m}t$. This may be derived by noting that in the co-moving frame, Δp equals $\dot{m}\Delta t v_E$, and also $(m - \dot{m}\Delta t)\Delta v$, or approximately $m\Delta v$. The same result is obtained in the stationary frame, where $\dot{m}\Delta t(v_E - v) = (m - \dot{m}\Delta t)(v + \Delta v)$.

To simplify integrating the equation, let $T = m_0/\dot{m}$, $A = \dot{m}v_E/m_0$, and $D = m_0 v_E/\dot{m}$; also let $\tau = t/T$. Then $m_0/m = 1/(1-\tau)$, and the equation may be written as $a/A = 1/(1-\tau)$. Assuming $v_0 = d_0 = 0$, this is easily integrated to yield $v/v_E = \ln(1/(1-\tau))$, and again to yield $d/D = (1-\tau)\ln(1-\tau) + \tau$; write the latter as δ. The work done on the payload equals the constant thrust $\dot{m}v_E$ times δD, or $m_0 v_E^2 \delta$.

In polar coordinates in two dimensions, the rocket equation is $m\ddot{r} - mr\dot{\theta}^2 = -\mu/r^2 + F_r$, $mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = rF_\theta$ where $F = \dot{m}v_E$ and F_r and F_θ are the radial and tangential components. Maximum work is extracted from the thrust when it is parallel to the velocity; in this case the work done is identical to the one dimensional case.

Letting $\mathcal{E}_0$ denote the specific energy in LEO, and $\mathcal{E}_1$ the specific energy in interplanetary orbit, we have $m\mathcal{E}_1 - m_0\mathcal{E}_0 = m_0 v_E^2 \delta$. We also have $\dot{m}v_E^2 = 2P$ where P is the propulsion power. From this,

$$t = \frac{\tau(m\mathcal{E}_1 - m_0\mathcal{E}_0)}{2\delta P}.$$

This can be minimized analytically. A computer minimization over quanta (2000 Kg, 50 KW per plant unit) yields the following; n is the number of quanta, n_p the number of plant, and t the time in days.

n	40	50	60	70	80
n_p	19	23	28	32	37
t	367	279	227	193	169

The time is the propulsion time; rough calculations show that taking in to account blocking by the Earth, the total time is larger by a factor of about 1.4.

With nuclear ion propulsion, many mirrors can be replaced by a small nuclear reactor. The advantage of nuclear power thus depends on the mirror weight, which should be as small as possible. As an example of the performance of higher specific power, if the power plant produces 75 W per Kg, with 20000 Kg plant and 22000 Kg fuel, the time is 235 days (of course, the plant mass represents a higher number of heat engines than at 2000 per unit).

6. Conclusion. Some simulations have been done which provide encouragement that the suggestions for a manned Mars program within an integrated space program, made in [Do02], are reasonable. Indeed, by using optimum trajectories a short stay mission with a total trip time of 280 days might be possible, and 300 days would be possible with reduced performance requirements. This type of mission might have a price tag as low as $25 billion.

Some comments on the relative merits of the short stay and long stay missions are as follows. First, either should be cast in the framework of an integrated program, in particular using as small a booster as possible and "batching" of fuel; and an ion engine slingshot for freight.

Batching requires solving the boiloff problem. What I call a "backbone" program has a maximum booster payload of 20 tons. Manned Mars is the only program where a larger payload would be convenient. A value of 40 tons is within the scope of the booster fleet design outlined in [Do02]. (and within the budget). Seven launches would fuel the outbound crew booster. At one launch per month, a boiloff rate of 1% is adequate. One way this might be achieved would be a removable (possibly active) outer cover.

For a long stay mission the outbound booster is smaller, but if batching is possible it would undoubtedly be cheaper than building a very large booster merely for this purpose. Note that other cases of the boiloff problem must be solved in either type of mission. The freight component on the other hand is much larger and more complex. This probably makes a long stay mission more expensive.

For Mars entry direct to the ground, it might be possible to fly an appropriate trajectory up to 1 revolution, to land at the desired spot. Another possibility is to go into orbit using an "outer" aeroshell, which is shed, and descent begins using an inner aeroshell.

Safety would seem to be comparable. The surface equipment can have greater redundancy in the long stay mission, but this is the lowest risk phase of either mission. The total time is less for the short stay, and the interplanetary flight time is less for the long stay.

The long stay is advantageous in that more human exploration is accomplished. However teleoperated probes are much cheaper and can explore a much larger area, so this advantage is marginal. The long stay is disadvantageous in that it is unknown what the stress on the astronauts would be, in particular what would be the best crew size.

Further discussion from [Do02] of one suggestion concerning the launch vehicles will be made here, namely, that the second stage and glider of the crew vehicle be vertically stacked. They could equally well be horizontally stacked, and there is a consideration which might decide in favor of this. With horizontal stacking the glider can be positioned so as not to interfere with the flow into the supersonic ramjet inlets. In particular, these can be positioned optimally, which may not be down as shown in figure 1 of [Do02]. The second stage cross section might be oval, flattened on the top, and its reentry surface can be on the bottom.

The inlets can be positioned forward in a cylindrical freight vehicle also. Access to the payload is achieved by completely separating the access section at the station. This method is available for the large object vechicle also.

Appendix 1. If $\mathbf{x}$ is a vector let x denote its magnitude $|\mathbf{x}| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$. With this convention the equation of motion for a body in orbit about the center of mass in a two body system is $\ddot{\mathbf{r}}+\mu\mathbf{r}/r^3 = 0$ where μ is the gravitational parameter $G(m_1+m_2)$ where G is the universal gravitational constant; μ may be approximated as Gm_1 when $m_2 \ll m_1$.

Let $\mathbf{v}$ denote $\dot{\mathbf{r}}$. Crossing the equation of motion with $\mathbf{r}$ yields that the angular momentum vector $\mathbf{r}\times\mathbf{v}$ is constant; this vector is denoted $\mathbf{h}$. The specific energy $\mathcal{E} = v^2/2 - \mu/r$ is also constant; this equation is called the energy equation.

Crossing the equation of motion with $\mathbf{h}$ on the right and using vector identites yields the equations

$$\mathbf{v}\times\mathbf{h} = \mu\big(\mathbf{e}+\frac{\mathbf{r}}{r}\big) \text{ and } r = \frac{p}{1+e\cos\nu}$$

where $\mathbf{e}$ (the eccentricity vector) is a constant of integration, e (the eccentricity) is the magnitude of $\mathbf{e}$, $\cos\nu = \mathbf{r}\cdot\mathbf{e}$ (ν is the true anomaly), and p (the semi-latus rectum) is defined as h^2/μ.

At periapsis $\nu = 0$, r equals $p/(1+e)$, and v equals $(1+e)\sqrt{\mu/p}$ (using $rv = h$). From this, $\mathcal{E} = -(1-e^2)\mu/(2p)$. From the energy equation, in general $v^2 = \mu(2/r - (1-e^2)/p)$. The cases $e = 0$ (circle), $0 < e < 1$ (ellipse), $e = 1$ (parabola), and $e > 1$ (hyperbola) for the form of the orbit may be distinguished. The foregoing derivation is well known and may be found in several sources, including [PC93].

Let $\alpha = 2\mathcal{E}$; since $v^2 = (\dot{r})^2 + (h/r)^2 = (\dot{r})^2 + \mu p/r^2$, from the energy equation

$$(r\dot{r})^2 = -\mu p + 2\mu r + \alpha r^2.$$

Define the "universal anomaly" (also called the regularizing variable) ψ up to an additive constant by the requirement $\dot{\psi} = 1/r$. Then

$$\frac{dr}{d\psi} = r\dot{r} = \mathbf{r} \cdot \mathbf{v}, \text{ and } \left(\frac{dr}{d\psi}\right)^2 = -\mu p + 2\mu r + \alpha r^2;$$

ψ is increasing in t, so $\psi = 0$ only at $t = t_0$. Let $\mathbf{r}_0$ and $\mathbf{v}_0$ be the values of $\mathbf{r}$ and $\mathbf{v}$ at time t_0.

This differential equation for $r(\psi)$ is solved as follows in [Go65]. Let

$$s_j(\psi) = \sum_i \frac{\alpha^i \psi^{2i+j}}{(2i+j)!}.$$

These functions are easily seen to satisfy

$$\frac{\partial s_j}{\partial \psi} = \begin{cases} \alpha s_1, & j = 0 \\ s_{j-1}, & j > 0 \end{cases}$$

$$s_j = \frac{\psi^j}{j!} + \alpha s_{j+2}$$

$$s_0^2 = \alpha s_1^2 + 1, \quad s_2(s_0 + 1) = s_1^2$$

The last may be seen by noting that $s_0(\psi) = \cosh(\sqrt{\alpha}\psi)$ and $s_1(\psi) = \sinh(\sqrt{\alpha}\psi)/\sqrt{\alpha}$, where for $x \geq 0$ $\sqrt{-x}$ is taken as $i\sqrt{x}$.

Using the identites for the s_i it is an algebraic exercise to verify that

$$r(\psi) = r_0 s_0 + \sigma_0 s_1 + \mu s_2$$

is a solution to the differential equation, where $\sigma_0 = \mathbf{r}_0 \cdot \mathbf{v}_0 = \sqrt{-\mu p + 2\mu r_0 + \alpha r_0^2}$. From $dt/d\psi = r$ it follows that

$$t(\psi) = t_0 + r_0 s_1 + \sigma_0 s_2 + \mu s_3.$$

The universal anomaly has many uses. For example a solution to the equation of motion may be obtained. Define $f = 1 - \mu s_2/r_0$ and $g = r_0 s_1 + \sigma_0 s_2$; then $\mathbf{r} = f\mathbf{r}_0 + g\mathbf{v}_0$ may be verified to be a solution. Noting that

$$\frac{d}{dt} = \frac{1}{r}\frac{d}{d\psi}$$

and $g = t - t_0 - \mu s_3$, differentiating yields $\mathbf{v} = \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0$ where $\dot{f} = -\mu s_1/(r r_0)$ and $\dot{g} = 1 - \mu s_2/r$.

The universal anomaly does not require distinguishing between elliptic and hyperbolic orbits. This allows specifying iterative algorithms which do not require distinguishing them, yielding better behavior for highly elliptical orbits for example. A Newtonian iteration for orbit reduction from three Earthbound observations is given in [Do88]. There is an error in the specification of the partial derivatives of the universal anomaly with respect to the orbital parameters given there; the correct values are as follows.

If t is considered as the independent variable the partial derivative ψ' of ψ with respect to some parameter is given by

$$\psi' = -\frac{t_0' + r_0' s_1 + \sigma_0' s_2 + \alpha'(r_0(\psi s_2 - s_3) + \sigma_0(\psi s_3 - 2s_4) + \mu(\psi s_4 - 3s_5))/2}{r}.$$

When r is the independent variable,

$$\psi' = -\frac{r_0' s_0 + \sigma_0' s_1 + \alpha'(r_0 \psi s_1 + \sigma_0(\psi s_2 - s_3) + \mu(\psi s_3 - 2s_4))/2}{\alpha r_0 s_1 + \sigma_0 s_0 + \mu s_1}.$$

Appendix 2. Suppose t_0 is a time of periapse, so $\sigma_0 = 0$ and $r_0 = p/(1+e)$. Let f' denote the partial derivative of f with respect to some parameter π. If f depends on π directly, and indirectly through ψ, then $f' = f'|_\psi + (\partial f/\partial\psi)\psi'$, where the first term is the partial derivative of f with ψ held fixed. Below f^* will be used to denote this. Considering t as the independent variable, $t' = 0$. Vector components are written $[x, y]$.

Unnormalized equations:

$r = r_0 s_0 + \mu s_2$

$t = t_0 + r_0 s_1 + \mu s_3$

$\partial s_0/\partial\psi = \alpha s_1$

$\partial s_i/\partial\psi = s_{i-1},\ i > 0$

$0 = t_0' + r_0' s_1 + r_0(s_0\psi' + s_1^*) + \mu(s_2\psi' + s_3^*)$

$\quad = r\psi' + t_0' + r_0' s_1 + r_0 s_1^* + \mu s_3^*$

$s_i^* = (\alpha'/2)(\psi s_{i+1} - i s_{i+2})$.

$(\partial t_0/\partial t_0) = 1$

$(\partial r_0/\partial p) = 1/(1+e)$

$(\partial r_0/\partial e) = -p/(1+e)^2$

$(\partial\alpha/\partial p) = \mu(1-e^2)/p^2$

$(\partial\alpha/\partial e) = 2\mu e/p$

$\mathbf{r} = [r_0 - \mu s_2, \sqrt{\mu p}\, s_1]$

Normalized equations, where $\rho_0 = 1/(1+e)$, $\beta = e^2 - 1$, $\tau = \sqrt{p^3/\mu}$:

$\rho = \rho_0 S_0 + S_2$

$t = t_0 + \tau(\rho_0 S_1 + S_3)$

$\partial S_0/\partial\Psi = \beta S_1$

$\partial S_i/\partial\Psi = S_{i-1},\ i > 0$

$0 = t_0' + \tau(\rho_0' S_1 + \rho_0(S_0\Psi' + S_1^*) + (S_2\Psi' + S_3^*)) + \tau'(\rho_0 S_1 + S_3)$

$$= \tau\rho\Psi' + t_0' + \tau(\rho_0' S_1 + \rho_0 S_1^* + S_3^*) + \tau'(\rho_0 S_1 + S_3)$$
$$S_i^* = (\beta'/2)(\Psi S_{i+1} - iS_{i+2})$$
$$\partial t_0/\partial t_0 = 1$$
$$\partial \rho_0/\partial e = -1/(1+e)^2$$
$$\partial \beta/\partial e = 2e$$
$$\partial \tau/\partial p = (3/2)\sqrt{p/\mu}$$
$$\boldsymbol{\rho} = [\rho_0 - S_2, S_1]$$
$$\boldsymbol{\rho}' = [\rho_0' - S_1\Psi' - S_2^*, S_0\Psi' + S_1^*]$$
$$\mathbf{r}' = p'\boldsymbol{\rho} + p\boldsymbol{\rho}'$$

$\mathbf{r}_S = M\mathbf{r}$ where $\mathbf{r}_S$ is the coordinates in space, $\mathbf{r}$ is the perifocal coordinates with z component set to 0, and

$$M = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{bmatrix}$$

(the last column is given for completeness, and can be deleted).

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