NEW CASES OF RECONSTRUCTIBILITY
OF SBT GRAPHS

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Abstract: In earlier papers the author showed that all graphs
which are not single-block trunk (SBT) graphs are reconstructible,
and that two families of SBT graphs are reconstructible. Here some
further families of SBT graphs are shown to be reconstructible.

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1. Introduction

Let $G$ be a graph. A limb in $G$ is a maximal induced subgraph,
which is a tree. If $G$ is not a tree each limb $L$ has a root $r$, which
attaches it to the rest of the graph. Removing the subgraphs $L - r$
for the limbs $L$, the remaining induced subgraph is called the trunk.

If $G$ is not a tree, and the trunk is inseparable, $G$ will be called
a single-block trunk (SBT) graph. In [2] it is shown that if $G$ is
not an SBT graph then $G$ is reconstructible. Let $n_v(G)$ ($n_e(G)$)
denote the number of vertices (edges) of $G$. If $G$ is an SBT graph
let $n_p(G) = n_e(G) - n_p(G)$; letting $B$ denote the block comprising
the trunk of $G$, this equals $n_e(B) - n_v(B)$. In [3] it is shown that
an SBT graph is reconstructible if $n_p \leq 1$. 
In this paper some results will be proved which imply that an SBT graph is reconstructible if \( n_p \leq 2 \). There are errors in [3], and new proofs will be given for \( n_p = 0 \) and \( n_p = 1 \) as well.

A graph \( G \) with a vertex \( v \) deleted is denoted \( G_v \).

It has long been known ([1]) that if a graph \( G \) has limbs, then the multiset of limbs with the attachment point marked, and the trunk, are reconstructible. By the size of a limb will be meant its number of edges, or the number of vertices other than the root. A limb of size 1 will be called a 1-limb.

The proof of lemma 11 of [2] is essentially a proof that the limbs are determined; it has an error, which will be corrected here. Namely, the claim is made that if in \( G_v \) for all degree 1 vertices \( v \) the limbs are \( n \) 1-lims, then in \( G \) the limbs are \( n + 1 \) 1-lims. This is true if \( n \geq 3 \); suppose \( n = 2 \). If there is a vertex \( w \) which is not of degree 2 such that in \( G_w \) has 2 components, 1 of which is an edge, then the limbs of \( G \) are a single path of length 2. If there is a vertex \( w \) which is not of degree 3 such that in \( G_w \) has 3 components, 2 of which are vertices, then the limbs of \( G \) are a single complete binary tree of height 1. If neither of these hold, the limbs are 2 1-lims.

Let \( L_{ds} \) be the multiset of limbs of size \( s \) attached to vertices of degree \( d \) in the trunk.

**Theorem 1.** The multiset of \( L_{ds} \) is reconstructible.

**Proof.** Except for the case of a single 1-limb, this is theorem 2 of [3]. In the case of a single 1-limb, \( G_v \) where \( v \) is the root of the 1-limb may be determined, and the degree of \( v \) is determined.

**Theorem 2.** Let \( G \) be an SBT graph. For given \( d \), unless the set of sizes of the limbs attached to degree \( d \) vertices of \( B \) is an interval \([0, m]\) for some \( m \), \( G \) is reconstructible.

**Proof.** If there is a limb of size \( s \) but none of size \( s - 1 \) let \( v \) be a leaf of a tree of size \( s \). In \( G_v \) there is a single limb of size \( s - 1 \), and this may be replaced by the limb missing from \( L_{ds} \).

**2. Automorphisms**
Suppose $G$ is an SBT graph and $\alpha$ is an automorphism of $B$. Let $G^\alpha$ be the result of $\alpha$’s action on $G$. That is, for a vertex $r$ of $B$, let $T_r$ (resp. $T^\alpha_r$) denote the tree rooted at $r$ in $G$ (resp. $G^\alpha$); then $T^\alpha_r = T_{\alpha(r)}$.

Suppose $v$ is a degree 1 vertex of a limb of $G$. $G$ “lies over” $G_v$, meaning that there is a vertex $r_0$ (namely the root of the limb containing $v$), such that $T_{rv}$ equals $T_r$, except if $r = r_0$ it has $v$ deleted. $G$ lies over $G_v$ in some other way iff $G^\alpha$ lies over $G_v$, where $\alpha$ is a nontrivial automorphism of $B$. Say that $\alpha$ is allowable (for $v$) in this case.

Clearly, an automorphism $\alpha$ of $G$ is allowable. Although they will not be used, some additional facts will be stated. Suppose $G^\alpha$ lies over $G_v$, where $r_1$ is the vertex of $B$ such that $T^\alpha_{r_1}$ does not equal $T_{r_1,v}$. It is possible that $r_1 = r_0$.

Suppose $r_0, \alpha(r_0), \ldots, \alpha^{l-1}(r_0)$ is the orbit of $r_0$. For $0 < i < l$, as long as $\alpha^i(r_0) \neq r_1$, $T_{\alpha^i(r_0)} = T_{\alpha^{i-1}(r_0)}$. This is so for all such $i$ iff $r_1 = r_0$. In any case, $r_1$ is in the orbit of $r_0$. For $r$ in some other orbit, $T_{\alpha(r)} = T_r$. $\alpha$ is an automorphism of $G$ iff $r_1 = r_0$. Otherwise, $T_{\alpha^i(r_0)} = T_{r_0,v}$ for $i \leq j < l$ where $\alpha^j(r_0) = r_1$.

**Theorem 3.** Suppose $G$ is an SBT graph. If $B$ is rigid and there are at least 2 degree 1 vertices then $G$ is reconstructible.

**Proof.** Let $v$ range over the degree 1 vertices. The $G_v$ may be “aligned” along $B$. $G$ is then the union of the $G_v$. \(\square\)

### 3. L-paths

Recalling some definitions of [4], an inseparable graph $A$ is said to be a subdivision of an inseparable graph $B$ if $A$ is obtained from $B$ by dividing edges by adding degree 2 vertices. Note that $n_p(A) = n_p(B)$. $B$ is said to be S-minimal if it is not a subdivision of another graph. It is shown in [4] that an inseparable graph $B$ is a subdivision of a unique S-minimal graph $B_r$.

If $G$ is a graph let $n^d_v(G)$ denote the number of vertices of degree $d$, and $n^{\geq d}_v(G)$ the number of degree at least $d$. For an inseparable graph $B$ $n^{\geq 3}_v(B) = n^{\geq 3}_v(B_r)$.

$n^{\geq 3}_v(B) = 0$ iff $B$ is a cycle iff $n_p(B) = 0$ iff $B_r$ is a 3-cycle. If $n^{\geq 3}_v(B) \neq 0$ then $n^{\geq 3}_v(B) \geq 2$. In this case, by a T-path in $B$ is
meant a maximal path all of whose interior vertices have degree 2
in $B$. A T-path in an SBT graph $G$ is a T-path in its trunk $B$.
An L-path in $G$ is a T-path, with the limbs attached to its interior
vertices adjoined, and its end points marked. The length of an L-
path $P$ is the length of its T-path, and the size of $P$ is the sum of
the sizes of the limbs, or the number of vertices not on the T-path.

An L-path $P$ of length $l$ may be oriented by labeling the vertices
of its T-path from 0 to $l$, from one end to the other. If $P$ is an
oriented L-path let $P^o$ denote $P$ with the opposite orientation, and
for a degree 1 vertex $v$ of a limb let $P_v$ have the inherited orientation.
Let $r_0 < \cdots r_t$ be the roots of the limbs, let $T_r$ denote the limb
rooted at vertex $r$, and let $T_{r,v}$ be the limb containing $v$. Let $\rho$
denote reversal of $P$. Say that $\rho$ is allowable if $P^o$ lies over $P_v$. If
$\rho$ is allowable say that it is type 1 if $T_{r,v}$ equals $T_{l-r_i}$, and type 2 if
$T_{r_i,v}$ equals $T_{l-r_i}$.

**Theorem 4.** Suppose $P$ is an L-path of size $s \geq 2$ and $v$ is a
degree 1 vertex of a limb. Suppose $\rho$ is allowable. If $\rho$ is type 1 then
$P^o = P$, otherwise $\rho$ is not allowable for $w$ where $w$ is the end of a
degree 1 vertex of $T_{l-r_i}$.

**Proof.** Straightforward.

4. $B$ a cycle

Suppose $G$ is an SBT graph and its trunk $B$ is a cycle.

**Lemma 5.** If $G$ has limbs of size $\geq 2$ then $G$ is reconstructible.

**Proof.** Let $u$ be a degree 1 vertex of a 1-limb $T$, and let $v$ be a
degree 1 vertex of a limb of size $\geq 2$. Let $R$ be $B$ with the roots
marked; this may be determined from $G_v$. Let $R_u$ be $B$ with the
roots marked, other than that of the limb $T$ containing $u$; this may
determined from $G_u$.

If there is a rotation which is an automorphism of $R$ let $\alpha$ be the
one of highest order. $G$ may be reconstructed from $G_v$, by locating
the orbit with $T$, and then the location of $T$, and adding a 1-limb.

If $R$ is rigid than $R$ fits over $R_u$ in only one way, whence the
location in $G_v$ of $T$ may be determined and $G$ reconstructed.
In the remaining case the automorphism group of \( R \) is generated by a single reflection, whose axis may be determined in any \( G_v \). \( G \) may be divided into two subgraphs \( P_1 \) and \( P_2 \), each containing the vertices on one side of the axis, where a vertex on the axis and its attached limb if any are in both \( P_i \). Let \( r \) denote the number of limbs of size \( \geq 2 \). If \( r \geq 3 \) then \( P_1 \) and \( P_2 \) in \( G \) may be determined by considering the \( G_v \).

Indeed, let \( r_1 \leq r_2 \) be the number of limbs on each \( P_i \). By considering the \( G_v \) these may be determined, and also the side \( P_2 \) with \( r_2 \) limbs. The other side \( P_1 \) is also easily determined, unless \( r_2 = r_1 + 1 \). In this case, let \( Q \) be the degree 1 vertices of limbs of size \( \geq 2 \) along \( P_2 \). Among the \( P_i \) (two each) of the \( G_v \) where \( v \in Q \), only one subgraph occurs \( |Q| \) times, and this is \( P_1 \).

In the remaining cases, \( r \leq 2 \); the reflection symmetry of \( R \) will not be used.

Suppose \( r = 2 \). Let \( T_1, T_2 \) be the two limbs of size \( \geq 2 \). By considering the \( G_u \) the distance from \( T_1 \) to \( T_2 \), and the number of 1-limbs in between, may be determined. If all 1-limbs are between, \( G \) may be reconstructed from \( G_v \) where \( v \) is a degree 1 vertex of \( T_i \) where the size of \( T_i \) is as small as possible.

Otherwise, the L-path \( P_0 \) between \( T_1 \) and \( T_2 \) may be determined from the \( G_u \). Let \( S \) be the 1-limb roots not on \( P_0 \). \( P_0 \) may be located in \( G_u \) for any \( u \) in a 1-limb with root in \( S \). For \( i = 1, 2 \) let \( P_i \) be the paths from the root of \( T_i \), away from \( P_0 \), containing the 1-limb roots closest to \( T_i \). For \( i = 1, 2, 3 \) let \( p_i \) be the number of 1-limbs along \( P_i \). Say that \( u \) is on \( P_i \) if its 1-limb is.

If \( p_0 \geq 2 \) \( G \) may be reconstructed from the \( G_u \) where \( u \) is on \( P_0 \), by theorem 4.

Suppose \( p_0 \leq 1 \). Suppose \( p_1 + p_2 \geq 3 \). By considering the \( G_u \) it may be determined if all of \( S \) lies along a single \( P_i \). If this is the case \( G \) may be reconstructed from the \( G_u \) for \( u \) in a 1-limb with root in \( S \), by taking the union. Otherwise, \( P_0 \cup P_1 \) and \( P_0 \cup P_2 \) may be determined, and \( G \) reconstructed from these. This is clear except for the case \( p_2 = p_1 + 1 \). In this case, among the paths (two each) of the \( G_u \) where \( u \) is on \( P_2 \), only one path occurs \( p_2 \) times, and this is \( P_1 \).

Suppose \( 1 \leq p_1 + p_2 \leq 2 \). If \( p_0 = 1 \) then \( P_1 \) and \( P_2 \) may be determined. There are 4 cases \( p_0, p_1 + p_2 \) as follows.

1.2. Whether the 1-limbs other than the one on \( P_0 \) are on the
same $P_i$ may be determined. $G$ may then be reconstructed as when $p_1 + p_2 \geq 3$.

1.1. Let $d_0$ be the distance from $T_1$ to $T_2$. In $G_v$ where $v$ is in the $T_i$ of least size, there is exactly 1 limb at distance $d$ from the $T_i$ of greatest size, with a 1-limb in between. $G$ is now readily reconstructed.

0.2. Let $d_0$ be the distance from $T_1$ to $T_2$, $d_1, d_2$ the distance from the 1-limbs to the nearest $T_i$, and $b$ the cycle length. The possibilities $d_0, d_1, d_2, b$ may be compared to the cyclic distances determined from $R$. Trying all 8 possible orderings of the second possibility, there is an ambiguity in only 3 cases, and in all of them $d_2 = (b - d_0)/2$. Whether this occurs is readily determined, and $G$ reconstructed in this case as well.

0.1. Let $d_0$ be the distance from $T_1$ to $T_2$. Suppose $v$ is in a $T_i$ of least size. If there is a single 1-limb at distance $d_0$ from $T_3-i$ in $T_v$ then $G$ may be reconstructed from $G_v$. Suppose there are 2 such. If the distance between them is $d_0$ $G$ may be reconstructed from $G_u$. Otherwise $G$ may be reconstructed from $G_v$ where $v$ is in $T_{3-i}$.

Suppose $r = 1$. Defined $P_1, P_2, p_1, p_2$ as in the case $r = 2$. If $p_1 + p_2 \geq 3$ $G$ may be reconstructed as in the case $r = 2$.

If $p_1 + p_2 = 2$ proceed as in the 0.2 subcase of the case $r = 2$. The possibilities are $d_1, d_2, d_1, b-d_2$ and $d_1, b-d_2, d_1, d_2$. Trying all 6 possible orderings of the second possibility, there is an ambiguity in only 2 cases, and in both of them $d_2 = b/2$. Whether this occurs is readily determined, and $G$ reconstructed in this case as well.

If $p_1 + p_2 = 1$ $G$ may be reconstructed from $G_v$. □

**Lemma 6.** Suppose the limbs of $G$ are 1 or more 1-limbs; then $G$ is reconstructible.

**Proof.** Let $b$ be the length of the cycle. Let $v$ range over the vertices of $B$. If for any $G_v$ the maximum path length is $b$ or $b - 2$, $G$ may be reconstructed from such a $G_v$. Otherwise, $b = 4t$ where $t > 0$, and numbering $B$ circularly from 0, there are 1-limbs attached to vertices numbered $i$ where $i \equiv 0, 1 \mod 4$. □

It is a question of interest whether, when there are at least 3 1-limbs, $G$ can be reconstructed from the $G_u$. The author has verified this for 3 or 4 1-limbs, and conjectures that it is true in general.
Theorem 7. If $B$ is a cycle then $G$ is reconstructible.

Proof. If there are no limbs the claim is readily verified. Otherwise it follows by lemmas 5 and 6.

\[5. \quad n_v^{\geq 3}(B) = 2\]

From hereon, for an SBT graph $G$ with trunk $B$, say that a vertex of $B$ of degree $\geq 3$ is an E-vertex. Let $n_v^E$ denote the number of them, i.e., $n_v^{\geq 3}(B)$. Call a limb attached to an E-vertex an E-limb. Call the remaining limbs I-limbs. A vertex $v$ will be said to be determining if $G_v$ can be determined, and $G$ reconstructed from $G_v$.

If $n_v^E(B) = 2$ then $n_p(B) \geq 1$ and $B$ consists of two vertices joined by $n_p(B) + 2$ T-paths, at most one of which is an edge; and $B_r$ consists of two vertices joined by an edge and $n_p(B) + 1$ T-paths of length 2. Let $d_E = n_p(B) + 2$ denote the E-vertex degree. A vertex $v$ is an E-vertex iff $G_v$ is acyclic.

Theorem 8. If $n_v^E = 2$ then $G$ is reconstructible.

Proof. Let $s_1 \leq s_2$ be the sizes of the E-limbs. If $\{s_1, s_2\}$ is not $\{0\}$ or $\{0, 1\}$ the claim follows by theorem 2. If the L-paths all have size 0 then an E-vertex with an E-limb if any attached is determining. Suppose neither of these cases holds. Say that an augmented L-path is an L-path, with the E-limb if any attached to its root. Let $u$ range over degree 1 vertices. The L-path sizes may be determined. If $v$ is the degree 1 vertex of an E-limb they may be determined from $G_v$; suppose there are no E-limbs, and let $n$ be the number of degree 1 vertices. If $n \geq 3$ the size multiset may be determined from those for the $G_u$, by a standard method (see e.g. the proof of lemma 11 of [2]). If $n = 1$, or if there is a single limb of size 2, the claim is trivial. If there are 2 1-limbs let $v$ be the root of a limb; the size list is $\{0^{d_E-2}, 1, 1\}$ if $G_v$ has a cycle with a limb, else $\{0^{d_E-1}, 2\}$.

Let $m$ be the minimum nonzero size, and let $r$ be the number of L-paths of size $m$. There is more than 1 limb iff either $r > 1$, or $r = 1$ and there are L-paths of size $> m$. It will next be shown that in this case, the multiset of augmented L-paths is determined.
Indeed, it is determined if a vertex $v$ is a non-root vertex of a limb of an L-path of size $m$; let $Q$ denote the set of these. Choose a $v \in Q$; the augmented L-paths of size $> m$ are those for $G_v$. If $r > 1$, consider the size $m$ augmented L-paths for the $G_u$, where $u$ ranges over $Q$. Each size $m$ augmented L-path appears $(r - 1)m$ times.

If $r = 1$ and there are L-paths of size greater than $m$, choose a vertex $v$ in the L-path of size $m$. Let $u$ range over the non-root limb vertices of the L-paths of size greater than $m$. For each such $u$, let $A_u$ be the multiset of size $m$ augmented L-paths in $G_u$. For each such $u$ we can also determine the multiset $B_u$ of size $m$ augmented L-paths in $G_{vu}$. The size $m$ augmented L-path $C$ of $G$ may be determined by taking the multiset union of the $A_u$ and of the $B_u$; the former will equal the latter, with some number of additional elements which are all copies of $C$.

Let $v$ be a degree 1 vertex of an L-path of size $m$. If $m > 1$ the size 0 augmented L-paths are those of $G_v$. Otherwise the augmented L-path $C$ of size 1 missing from $G$ is determined, and the length 0 augmented L-paths of $G$ are those of $G_v$, with $C_v$ deleted.

If there is an E-limb, $G$ is determined by its multiset of augmented L-paths; suppose not.

Suppose there is an $s \geq 2$ such that there is an L-path $P$ of size $s$ but none of size $s - 1$. Then it is determined whether $v$ is a degree 1 vertex of a limb of $P$, and if so $P_v$ may be found in $G_v$. $G$ may be reconstructed using theorem 4.

Suppose $d_E \geq 4$. Since there are I-limbs there must be an L-path of size 1; the root of its 1-limb is determining.

Suppose there is a single L-path $P$ of size $m \geq 2$, and there is an E-vertex. Then $P$ may be found in $G_v$ where $v$ is the degree 1 vertex of the E-limb. $G$ may be reconstructed using theorem 4.

If $P$ has a single limb of size $\geq 2$ a degree 1 vertex of the limb is determining.

Suppose there is a single L-path $P$ of size $m \geq 3$, and there is no E-limb. Let $P_0$ be the sub-path of $P$ containing the limbs, with limbs at the ends. The length of $P_0$ may be determined from the $G_v$ where $v$ ranges over the degree 1 vertices. $P_0$ may then be determined from $G_w$ where $w$ is a degree 2 vertex on a size 0 L-path of length $\geq 2$. $G$ may be reconstructed using theorem 4 applied to $P_0$. 
The remaining cases are as follows, where $ijk$ are the L-path sizes, and E indicates an E-limb also.

002. 2 1-limbs. Let $v$ be a vertex such that $G_v$ is a cycle with a single limb attached. Since the lengths of the T-paths are known the length of the T-path $P$ is determined from the length of the cycles; and also the location on the cycle of the other E-vertex. Since it is known whether $v$ has a 1-limb attached, $G$ may be reconstructed from $G_v$.

001. Let $v$ be a vertex such that $G_v$ is a cycle with a single limb attached, or no limbs. The argument is similar to the preceding case; if there is no 1-limb the E-vertices may be spaced on the cycle.

001E. Let $v$ be a vertex such that $G_v$ is a cycle with a single limb attached. The argument is similar to the preceding case.

In the remaining cases, $d_E = 3$ and the L-paths are determined. If an L-path of length 2 has a limb let $s$ be the smallest size of such; an end vertex of a size $s$ limb on an L-path of length 2 is determining. Thus it may be assumed that the L-paths have length $\geq 3$.

A vertex $u$ will be specified, as that where $G_u$ has a cycle with certain limbs. One of these will be recognizable as having an E-vertex $x$ as its root. Let $P_1$ be the path containing $u$, and $l_1$ its length; $l_1$ equals the sum of the lengths of the T-paths, minus the length the cycle. Knowing $l_1$ and the size of $P_1$, it may be reconstructed by connecting an end of a new edge up to a uniquely determined vertex of the limb rooted at $x$, and possibly also an isolated vertex. Knowing $P_1$ the other two paths $P_2$ and $P_3$ are known, and the other E-vertex $y$ can be located on the cycle. $G$ may be reconstructed by attaching the other end of the new edge of $P_1$ to the cycle at $y$. Let $l_2, l_3$ be the lengths of $P_2, P_3$; $y$ may be either at distance $l_2$ from $x$ counterclockwise around the cycle, or at distance $l_3$.

011. Let $u$ be such that $G_u$ has 1 1-limb and a limb of size $\geq 2$, $x$ being the root of the latter. Let $P_2$ be the size 0 path, let $d, d'$ be the distances from the root of the 1-limb of $P_3$ to the ends, and let $e$ be the distance from $x$ on the cycle in $G_u$, to the 1-limb on the cycle. There are two possibilities for the location of the other E-vertex in $G_u$; for both possibilities for $y$ to be allowable, $l_2 < e$ must hold, and both $e$ and $e - l_2$ must be among $\{d, d'\}$; it follows that
e = (l_2 + l_3)/2 must hold, and the two possibilities are isomorphic.

012. Let u be such that $G_u$ has 1 1-limb and a 1 limb of size \( \geq 2 \), $x$ being the root of the latter. This case differs from case 011 only in that either the limb at $x$ is of size \( \geq 3 \), or there is an isolated vertex.

111. Let u be such that $G_u$ has 2 1-limb and a 1 limb of size \( \geq 2 \), $x$ being the root of the latter. Let $e, e'$ be the circular distance from the roots $r, r'$ of the 1-limbs of $G_u$ to $x$, along the arc not containing the other 1-limb. For a possibility for $y$ to be allowed, it must lie on the arc between $r$ and $r'$, not containing $x$. If $e = e'$ the possibilities yield isomorphic graphs. If either $P_2$ or $P_3$ is symmetric the degree 1 vertex of its 1-limb is determining. Otherwise an ambiguity can occur only if $l_3 - e = e'$ and $l_2 - e' = e$, whence $l_2 = l_3$ and $y$ is determined.

112. Let u be such that $G_u$ has 2 1-limbs and a 1 limb of size \( \geq 2 \), $x$ being the root of the latter. This case differs from case 111 only as case 012 differs from case 011.

122. Let u be such that $G_u$ has 3 1-limbs and a 1 limb of size \( \geq 2 \), or 1 1-limb, 1 limb of size 2, and 1 limb of size \( \geq 3 \), $x$ being the root of the largest. In the latter case the size 2 path is readily located on the cycle. In the former let $P_2$ be the L-path with 2 1-limbs and $P_3$ the L-path with 1 1-limb. Write $l_2 = b_1 + b_2 + b_3$ where $b_1, b_3$ are the distances from the roots of the 1-limbs to the ends and $b_2$ is the distance between them. Similarly write $l_3 = a_1 + a_2$. One direction of traversing the cycle may be written (renumbering as necessary) as $a_1, a_2, b_1, b_2, b_3$. An ambiguity can occur only if this sequence equals one of 4 possibilities with the $b$’s first. These may be seen to all be impossible.

123. Let $P_1$ be as in case 122; the argument differs from case 122 only in how $P_1$ is determined. It may be assumed that there is at most one isolated vertex in $G_u$.

\[\square\]

6. L-multipaths

An L-multipath in an SBT graph $G$ is defined to be the subgraph consisting of all L-paths between a pair of E-vertices. The number of L-paths is called the multiplicity. The size of an L-multipath is the sum of the sizes of its limbs.
If $G$ is a graph with no degree 1 vertices let $\text{MPG}(G)$ (the “multipath graph” of $G$) be the graph whose vertices are the $E$-vertices of $G$; two such are joined by an edge iff they are joined by one or more T-paths in $G$. If $B$ is an inseparable graph, $\text{MPG}(B)$ equals $\text{MPG}(B_r)$. If $B$ is an S-minimal inseparable graph then $\text{MPG}(B)$ is obtained from $B$ by deleting all T-paths of length 2. For an inseparable graph $B$, write $B_m$ for $\text{MPG}(B)$. $B_m$ is inseparable, and any inseparable graph may occur as $B_m$ (replace each edge by a triangle, or add 2 length 2 paths to an edge).

Note that an automorphism of $B$ acts on $B_m$.

For an inseparable graph $B$, $B_m$ is an edge iff $n^3_v(B) = 2$. Otherwise $B_m$ has no degree 1 vertices, else $B$ would be separable. When $n_p = 1$ then $B$ is an edge. By theorem 8, $G$ is reconstructible in this case.

An L-multipath may be oriented by labeling the vertices of each T-path consecutively starting at an E-vertex. If $M$ is an oriented L-multipath let $M^r$ denote $M$ with the opposite orientation; and for a degree 1 vertex $v$ of a limb let $M_v$ have the inherited orientation.

**Theorem 9.** Suppose $M$ is an oriented L-multipath of size $s \geq 2$, and for every degree 1 vertex $v$ of a limb, $M_v$ is an oriented subgraph of $M^r$. Then $M = M^r$.

**Proof.** If there is only one path with limbs the claim follows by theorem 4; suppose there is more than 1. Let $m$ be the smallest nonzero size of an L-path. Let $P_j$ for $1 \leq j \leq k$ be the L-paths. Let $v$ be a degree 1 vertex of a limb on a path $P_i$ of size $m$. Let $\pi$ be a permutation of $[1, k]$ such that $P_{jv} \subseteq P_{\pi(j)}^r$ for all $j$. Let $S$ be the indices of the L-paths of size $> m$; then $\pi[S] = S$. Thus, for any $j \in S$ there is a cycle $j, \pi(j), \ldots, \pi^{t-1}(j), \pi^t(j) = j$ such that $P_{\pi^r(j)} \subseteq P_{\pi^{r+s+1}(j)}$ for $1 \leq s < t$. It follows that $P_{\pi^r(j)} = P_{\pi^{r+s+1}(j)}$ for $1 \leq s < t$, and so all $P_{\pi^r(j)}$ are equal, or $t$ is even and $P_{\pi^r(j)}$ equals $P_j$ for $t/2$ values of $s$, and $P_j^r$ for $t/2$ values of $s$. In either case, the multiset of L-paths of size $> m$ is symmetric.

Let $T$ be the indices of the size $m$ L-paths. Renumbering, $v$ is in $P_0$ and one of the cycles of $\pi$ is 0, 1, \ldots, $t - 1$. The L-path multiset of each other cycle is symmetric. In the 0 cycle, $P_0$ appears $\lceil (t+1)/2 \rceil$ times and $P_0^r$ appears $\lfloor (t+1)/2 \rfloor$ times. If $t$ is even or $P_0$ is symmetric then $M$ is symmetric. If all L-paths have size $m$ $P_0$ must be symmetric.
If there is an L-path of size $\geq m + 2$, let $v$ be a vertex of such. Let $\pi$ be the permutation for $G_v$. Then $\pi[T] = T$, so the L-path multiset for $T$ is symmetric, so $M$ is symmetric. In the remaining case there is an L-path of size $m + 1$, and for any such $Q$, and any vertex $v$ of a limb of $Q, Q_v = P_0^v$; this is clearly impossible.

**Theorem 10.** Let $G$ be an SBT graph with $v^E_v(B) \geq 3$.

a. If $G$ has limbs the L-multipath sizes are determined.

b. If $G$ has more than one L-multipath of nonzero size then the L-multipaths are determined.

Suppose the L-multipaths have been determined.

c. If there is an L-multipath of size $s \geq 2$ but none of size $s - 1$, $G$ is determined.

d. For each multiplicity $m$, if there is an L-multipath of multiplicity $m$ and size $s \geq 2$, but none of multiplicity $m$ and size $s - 1$, then $G$ is determined.

**Proof.** Part a is proved similarly to the claim for L-paths in theorem 8. Indeed, the argument for the L-multipath sizes is virtually identical, except for the case of 2 1-limbs. Let $x, y$ be the degree 1 vertices. If the multipaths with an I-limb attached in $G_x$ or $G_y$ differ, or if they are both a path of length 2, then there are 2 size 1 L-multipaths. If they are a path of length $\geq 3$, there is a single size 2 L-multipath iff there is a vertex $w$ such that in $G_w$ there is a single limb. If they are a multipath of multiplicity $\geq 2$, there is a single size 2 L-multipath iff there is a vertex $w$ on a path of a multipath of multiplicity $\geq 2$, such that there is an L-multipath with 2 I-limbs of size 1.

The argument for part b is readily adapted from an argument in the proof of theorem 8.

For part c, let $v$ range over degree 1 vertices of limbs of the L-multipaths of size $s$. For each $v$ both the L-multipath $M$ containing $v$, and $M_v$, are determined. If for some $v M_v$ is not a subgraph of $M^r$, $M$ may be laid over $M_v$. Otherwise, by theorem 9, for any $v$, $M_v$ may be replaced by $M$.

Part d follows as part c, letting $v$ range over degree 1 vertices of L-multipaths of multiplicity $m$ and size $s$.

7. $B_m$ a cycle
Lemma 11. Suppose $G$ is an inseparable graph. It is determined whether $B_m$ is a cycle.

Proof. There must be $\geq 3$ E-vertices. $B_m$ is a cycle iff, for any E-vertex $v$, $\text{MPG}(G^T_v)$ is a path, where for a graph $G$, $G^T$ denotes the trunk of $G$. \qed

Suppose for the rest of the section that $G$ is an SBT graph with $B_m$ a cycle. Say that a multipath is fat (resp. thin) if it has multiplicity $> 1$ (resp. 1). Say that it is $T_i$ if it is thin and has length $i$; and $F_{ij}$ for $1 \leq i \leq j$ if it has multiplicity 2 and path lengths $i, j$. The same terminology applies to L-multipaths.

The automorphism group of $B$ is either trivial, generated by a rotation, generated by a reflection, or generated by a reflection and a rotation. In cases 2 and 4 say that $B$ admits a rotation. If $B$ admits a rotation then there is a subgraph $Q$ of $B$, which is a concatenation of $q$ multipaths, such that $B$ is the circular concatenation of $q'$ copies of $Q$. Further, if $q$ is as small as possible then for any $Q$, these are the only copies of $Q$ around $B$.

Lemma 12. If $G$ has no limbs then $G$ is reconstructible.

Proof. It may be assumed that all $T$-paths have length at most 3, since if $v$ is a degree 2 vertex of distance $\geq 2$ from both ends then $G$ may be reconstructed from $G_v$.

If there are any $T_i$ multipaths for $i = 2, 3$ let $w$ be an interior vertex of such; $G$ is determined from $G_w$, except that an end cycle of $G_w$ of length 4 might be either an $F_{13}$ or $F_{22}$ multipath. It is determined whether a vertex $v$ is an interior vertex of the path of length 3 of an $F_{13}$ multipath, and so the number of $F_{13}$ multipaths is determined. It follows that if $w$ is an interior vertex of a $T_3$ multipath then $G$ is determined from $G_w$.

Suppose there are no $T_3$ multipaths but there are $T_2$ multipaths. Let $w$ be the interior vertex of a $T_2$ multipath $P$. Unless one end of $P$ belongs to an $F_{13}$ multipath and the other to an $F_{22}$ multipath, $G$ is determined from $G_w$. In particular it may be assumed that there is a degree 3 vertex $v$ such that in $G_v$, one end is a 3 vertex complete binary tree and the other is a path of length 1; $G$ may be reconstructed from $G_v$.

Suppose there are no $T_2$ or $T_3$ multipaths, and there are $T_1$ multipaths. A vertex $v$ is the end vertex of an $T_1$ path iff one
end $D$ of $G_v$ is a fat multipath or a cycle; let $E$ be the other end, which is a tree. The multipath from which $E$ is derived may be determined. This is clear if $E$ is not a path, or if $E$ is a path of length 1 or 3. If $E$ is a path of length 2 the choice between a $T_1$ multipath followed by an $F_{12}$ multipath, and an $F_{13}$ multipath, may be made on the basis of the number of $T_1$ multipaths. If $D$ is not a cycle of length 4 $G$ is readily determined. If $D$ is a cycle of length 4 $G$ is determined also, since the number of $F_{13}$ multipaths is known.

Letting $v$ be the other end of the $T_1$ multipath, $E$ must be an $F_{13}$ also. Thus, every $T_1$ multipath must have its ends in an $F_{13}$ multipath, and so $G$ may be reconstructed from $G_v$ where $v$ is an end of an $T_1$ multipath.

Suppose all multipaths are fat. Let $v$ be an E-vertex. Add a new vertex to $G_v$ and connect it to the degree 1 vertices. If either 0 or 2 additional edges are required, these may be added as needed to reconstruct $G$ from $G_v$. Otherwise, in every adjacent pair of multipaths, one contains a length 1 path and the other does not. Which root of the trees in $G_v$ to connect to the new vertex is readily determined in $G_v$.

**Lemma 13.** Let $q$ be the number of limbs of size $\geq 2$. If $q \geq 2$ then $G$ is reconstructible.

**Proof.** Let $u$ be a degree 1 vertex of a 1-limb $T$, and let $v$ be a degree 1 vertex of a limb of size $\geq 2$. Let $R$ be $B$ with the limb roots marked, and let $R_u$ be $B$ with the roots marked, other than that of $T$.

The cases where $R$ has a rotation which is an automorphism, and where $R$ is rigid, may be proved as in the case of lemma 5.

In the remaining case, the automorphism group of $R$ is generated by a single reflection, and the axis may be determined in any $G_v$. $G$ may be divided into two subgraphs $P_1$ and $P_2$, each containing the vertices on one side of the axis, where a vertex on the axis and its attached limb if any are in both $P_i$.

If $q \geq 3$ then $P_1$ and $P_2$ in $G$ may be determined by considering the $G_v$, as in the proof of lemma 5.

Suppose $q = 2$. The roots $r_1$ and $r_2$ of the size $\geq 2$ limbs may be located in $P_1$, so it only remains to determine if in $G$ they are located in opposite sides.
If \( r_1 = r_2 \) then they are on opposite side. If one of \( r_1, r_2 \) is on the axis the question is irrelevant. If one of the limbs (say the one at \( r_1 \)) has size 3 then this can be noted in the marked positions in \( P_1 \); whether \( r_1, r_2 \) are on the same side can then be determined from \( G_v \) where \( v \) is a degree 1 vertex of the limb of size 3.

Suppose \( r_1, r_2 \) are in the same L-multipath \( M \) in \( P_1 \). It may be determined whether they are in the same multipath in \( G \) from any \( G_u \). Unless \( M \) is on the axis, \( G \) may clearly be reconstructed. If \( M \) is on the axis, the axis is that of \( M \) with its limbs removed, so whether \( r_1, r_2 \) are on the same side is determined from any \( G_u \).

Suppose there is no L-multipath containing \( r_1, r_2 \).

Suppose there are L-multipaths \( M_1, M_2 \) on the axis with \( r_1 \) in \( M_1 \) and \( r_2 \) in \( M_2 \). If either limb is an I-limb, the axis is known. Otherwise, the distance \( d \) between \( r_1 \) and \( r_2 \) in \( B_m \) can be determined from any \( G_u \); \( r_1, r_2 \) are on opposite sides iff \( d = l/2 \) where \( l \) is the length of \( B_m \).

Suppose \( r_1 \) is on \( M_1 \) where \( M_1 \) is on the axis, but \( r_2 \) is not on the other L-multipath on the axis, if any. In \( G_u \), let \( w \) be the E-vertex of \( M_1 \) closest to \( r_2 \), and let \( P \) be the T-path of \( M_1 \) containing \( r_2 \), and let \( l \) be its length. Then \( r_1 \) and \( r_2 \) are on the same size iff the distance from \( r_1 \) to \( w \) is less than \( l/2 \).

In the remaining case, let \( C \) be a cycle of length \( 2l \), with alternate positions denoting E-vertices and L-multipaths. Number the nodes counterclockwise from 0, where node 0 is an E-vertex or L-multipath according to which lies at the top of the axis. The locations of \( r_1 \) and \( r_2 \) on one side of \( C \) are determined; let these be \( 0 < i < j < l \). If \( r_1, r_2 \) are on the same side the distance between them in \( C \) is \( j - i \). If they are on opposite sides, if \( i + j \leq l \) the distance is \( i + j \), and if \( i + j > l \) the distance is \( 2l - i - j \); in either case it may be determined if \( r_1, r_2 \) are on the same side.

In the remaining cases, \( B \) is known.

**Lemma 14.** Suppose \( G \) has 1 limb; then \( G \) is reconstructible.

**Proof.** By theorem 2 the limb may be assumed to be a 1-limb. Let \( x \) be its root.

Suppose the limb is an I-limb. The L-multipath \( M \) containing \( x \) may be determined from \( G_x \). Indeed, the multipath is known from \( B \), and the location of \( x \) may be determined from the limbs of \( G_x \).
Suppose there is a thin L-multipath $P$ of length $\geq 4$. Let $z$ be an interior vertex of $P$ which is at distance 2 from an end whose neighbor interior vertex has no limb. $G$ may be readily reconstructed from $G_z$, no matter where $x$ is located.

If $M$ is a T$_3$ L-multipath there is a vertex $y$ such that in $G_y$ one end $E_1$ is a multipath with a path of length 2 attached, and the other end $E_2$ is a multipath or cycle. The multipath $M_1$ and $M_2$ giving rise to $E_1$ and $E_2$ may be determined, and $G$ reconstructed from $G_y$.

If $M$ is a T$_2$ L-multipath the ends $E_1$ and $E_2$ in $G_x$ are each a multipath or cycle. Unless both are 4-cycles, $G$ is readily reconstructed; this is also true if the multipaths missing from $B$ are both F$_{22}$ or F$_{13}$ multipaths. In the remaining case there is a vertex $y$ such that in $G_y$ there is a path of length $\geq 4$, with an F$_{13}$ multipath at one end with a 1-limb attached to the closest vertex to that end of the path; $G$ is readily reconstructed from $G_y$.

In the remaining cases $M$ is a fat multipath.

If $M$ is adjacent to a T$_3$ L-multipath there is a vertex $y$ such that in $G_y$ one end $E_1$ is a cycle or multipath with 2 1-limbs attached, and the other end $E_2$ is a cycle or multipath. $E_1$ is derived from $M$, so the multipath from which $E_2$ is derived is also known. If $E_1$ is a multipath $G$ is readily reconstructed from $G_u$. Otherwise $M$ is an $F_{ij}$ L-multipath. If $i = j$ $G$ is readily reconstructed, so suppose $i < j$. Let $l_1 \leq l_2$ be the distances from the attachment vertex of $E_1$ to the roots of the limbs; if $l_1 = l_2$ $G$ is readily reconstructed, so suppose $l_1 < l_2$. It is readily verified that if the 1-limb is on the path of length $i$ then $j = l_2$, and if it is on the path of length $j$ then $i = l_1$; $G$ may thus be reconstructed.

If $M$ is adjacent to a T$_2$ L-multipath there is a vertex $y$ such that in $G_y$ one end $E_1$ is a cycle or multipath with a 1-limb attached, and the other end $E_2$ is a cycle or multipath. $E_1$ may be replaced by $M$. The multipath missing from $B$ is then known, and $G$ may be reconstructed.

If $M$ is adjacent to a T$_1$ L-multipath there is a vertex $y$ such that in $G_y$ one end $E_1$ is a cycle or multipath with a 1-limb attached, and the other end $E_2$ is a tree. $E_1$ may be replaced by $M$. The multipath $M_2$ missing from $B$ may then be determined, and also the part of it to be replaced may be determined and $G$ reconstructed.

Suppose there is a vertex $y$ such that in $G_y$ there is an L-
multipath with an I-limb and an E-limb (both 1-limbs). Since the missing multipath is known from $B$, a new vertex may be connected to the end of the E-limb, and to a second vertex, to reconstruct $G$ from $G_y$.

If there is no such $y$, let $w$ be such that in $G_w$ the number of length 2 paths in a multipath adjacent to $M$ is as small as possible; $G$ may be reconstructed from $G_w$.

Now suppose that the limb is an E-limb, with root $v$. At least 1 multipath $M$ incident to $v$ is fat. If the other $T$ is thin, the argument is similar to arguments already given. Namely, if $T$ has length $\geq 4$ let $u$ on $T$ have distance at least 2 from both ends; if $T$ has length 3 let $u$ such that $E_1$ in $G_u$ has a limb of size 2; if $T$ has size 2 let $u$ be such that $E_1$ has a 1-limb; and if $T$ has length 1 let $u$ be such that $E_1$ has a 1-limb.

In the remaining case, the other multipath incident to $v$ is also fat. If one of $M, N$ has a path of length 3, suppose w.l.g. that $M$ is one of smallest multiplicity. Let $u$ be on $M$ such that in $G_u$ there is a single limb, of size $\geq 2$. Unless $M_u = N G$ may be reconstructed from $G_u$. If $N = M_u$ $G$ is reconstructible from $G_w$ where $w$ is on a length 2 path of $M$. If neither $M, N$ has a path of length 3 suppose w.l.g. that $M$ has smallest multiplicity between $M, N$; $G$ is reconstructible from $G_w$ where $w$ is on a length 2 path of $M$.

**Lemma 15.** Suppose the limbs of $G$ are 2 1-limbs, which do not occur in a common L-multipath; then $G$ is reconstructible.

**Proof.** The L-multipaths may be determined from the $G_u$ for $u$ a vertex of degree 1.

Suppose $M$ is an L-multipath of multiplicity $\geq 2$ containing a path $P$ with an E-limb.

Suppose $M$ has a T-path $Q$ other than $P$, of length $\geq 3$. Let $v$ be an interior vertex of $Q$. In $G_v$ there is an E-vertex $w$ such that $w$ has a limb, and an incident L-multipath with an I-limb; further there is such a $v$ with only one incident L-multipath with an I-limb. $G$ may be reconstructed from $G_v$.

Otherwise in all such $M$ all paths other than $P$ have length $\leq 2$. Letting $v$ be the interior vertex of a path of length 2 other than $P$, $G$ may be readily reconstructed, except in the case where $M$ has multiplicity 2 and either is the only $M$, or the other one is also has multiplicity 2, or the other one is $M$ with a length 2
path added. In these cases let $v$ be an E-vertex of $M$. If the other L-multipath incident to $v$ has an I-limb $G$ is readily reconstructed from $G_v$. If $P$ has length $\geq 3 v$ may be chosen so that $G$ is readily reconstructed from $G_v$. In the remaining cases the multipath of $M$ is either $F_{21}$ or $F_{22}$, and $G$ may be reconstructed from $G_v$ unless the other multipath incident to $v$ is $F_{31}$ or $F_{32}$ respectively. This must be the case for any $v$. Let $w$ be the vertex on the path of length $3$, not adjacent to $v$; $G$ may be reconstructed from $G_v$.

Suppose $v$ is the root of an I-limb on a $T_i$ path $P$ where $i \geq 3$. By hypothesis $P$ has no other limb attached. One of the ends of $G_v$ is a multipath with a path attached, so its multipath is known, so as usual $G$ is reconstructible from $G_v$.

Suppose $v$ is the root of an I-limb on a $T_2$ path $P$. $G$ is reconstructible from $G_v$, unless both ends of $G_v$ are cycles of the same size $i$. If both missing multipaths are $F_{i1}$ for some $i$ then again $G$ is reconstructible from $G_v$. If either missing multipath is not $F_{i1}$ for some $i$, Otherwise an E-vertex $w$ incident to an $F_{ij}$ multipath where $j \geq 2$ and $P$ may be found, and $G$ reconstructed from $G_w$.

Suppose both limbs are E-limbs.

If $M$ is a thin L-multipath of length $\geq 2$ with an E-limb at one end let $v$ be an interior vertex of $M$; $G$ is reconstructible from $G_v$.

Suppose $M$ has a T-path $P$ of length $\geq 3$. Let $w$ be the interior vertex of $P$ adjacent to $v$. $G$ is reconstructible from $G_w$, except in the following case. $M$ is an $F_{31}$ multipath and there is a fat multipath $A$ such that there is a sequence of adjacent L-multipaths $AEMA$ where $E$ is a $T_1$ L-multipath, $v$ is the common vertex of $MA$, and there is a 1-limbs attached to the $AE$ common vertex. In this case $G$ is reconstructible from $G_u$ where $u$ is the degree 1 vertex of the limb with root $v$.

In the remaining case all multipaths adjacent to a root $v$ of a limb are $F_{2m_0}$ or $F_{2m_1}$ where $m \geq 2$, $F_{21}$, or $T_1$. Unless the multipaths at $v$ are $F_{2m_0}$ and $F_{2m_1}$ where $m \geq 2 G$ is reconstructible from $G_v$. In the exceptional case $G$ is reconstructible from $G_y$ where $y$ is the interior vertex of a length 2 T-path of an $F_{m1}$ multipath.  

**Lemma 16.** Suppose $B$ admits a rotation; then $G$ is reconstructible.
Proof. Let $x$ be a vertex on an L-path of an L-multipath $M$ of multiplicity $\geq 2$ in $G$. With $q$ as at the start of the section, let $Q$ be a run of $q$ multipaths, not including the multipath $M_0$ missing from $G_x$. Using $Q$, the location of $M_x$ in $G_x$ may be found. $x$ may be chosen so that the E-limbs of $G_x$ are those of $G$, with one of them having an extra subtree $A$ of its root; further the root is an E-vertex of $M$.

$G$ may be reconstructed from $G_x$ by adding a new vertex $x$, connecting it to the E-vertex of $M_x$ other than the root of $A$, connecting it as necessary to any disconnected component(s), and connecting it to the correct vertex of $A$. The latter may be determined, since the length of the L-path containing $x$ is known, and I-limbs are 1-limbs, with the possible exception of a single I-limb of size 2.

Lemma 17. Suppose the automorphism group of $B$ is generated by a reflection; then $G$ is reconstructible.

Proof. Similarly to the proof of lemma 13 when the automorphism group of $R$ is generated by a single reflection, the axis may be determined in any $G_u$ where $u$ is a degree 1 vertex. $G$ may be divided into two parts $P_1$ and $P_2$. If the number $q$ of limbs is $\geq 3$ $G$ may be reconstructed as in lemma 13.

Suppose $q = 2$. The roots $r_1$ and $r_2$ of the limbs may be located in $P_1$, so it only remains to determine if in $G$ they are located in opposite sides.

If $r_1 = r_2$ then they are on opposite side. If one of $r_1, r_2$ is on the axis the question is irrelevant. If one of the limbs (say the one at $r_1$) has size 2 then this can be noted in the marked positions in $P_1$; whether $r_1, r_2$ are on the same side can then be determined from $G_v$ where $v$ is a degree 1 vertex of the limb of size 2.

Suppose $r_1, r_2$ are in the same L-multipath $M$ in $P_1$. Suppose $M$ is thin. If $M$ is on the axis there is a vertex $w$ on the path of $M$ such that in $G_w$ there is only one limb, and $G$ may be reconstructed from $G_w$. If $M$ is not on the axis there is a vertex $w$ on the path of $M$ such that in $G_w$ there is only one limb, iff $r_1, r_2$ are on the same side. Suppose $M$ is fat. If $M$ is on the axis let $w$ be the root of an I-limb of $M$; $G$ may be reconstructed from $G_w$. If $M$ is not on the axis let $w$ be the root of an I-limb of $M$ if there is one, else any vertex on a T-path of $M$; it can be determined from $G_w$ whether $r_1, r_2$ are on the same side.
If there is no L-multipath containing \( r_1, r_2 \) the claim follows by lemma 15.

\( \square \)

**Theorem 18.** If \( B_m \) is a cycle then \( G \) is reconstructible.

**Proof.** This follows by lemmas 12 to 17, and theorem 3. \( \square \)

8. \( B_m K_4 \)

In this section, \( G \) will denote an SBT graph with \( B_m \) equaling \( K_4 \).

**Lemma 19.** If \( G \) has no limbs then \( G \) is reconstructible.

**Proof.** If for some \( m \geq 2 \) there is a multipath of multiplicity \( m \) but none of multiplicity \( m - 1 \), let \( v \) be a vertex on a path of such a multipath; then \( G \) is reconstructible from \( G_v \). Thus, the multipath multiplicities may be assumed to form an interval \([1, m]\).

Suppose all thin multipaths are edges, and there is at least 1 fat multipath. Let \( m = 2 \). If a multipath \( M \) of multiplicity \( m \) has all paths of length \( \geq 2 \) let \( v \) be an interior vertex of a path of \( M \); \( G \) is reconstructible from \( G_v \). Proceeding inductively on \( m \) and using the same argument, every fat multipath must contain an edge. It now follows that \( G \) is reconstructible from \( G_v \) where \( v \) is any E-vertex.

Suppose there is a thin multipath \( P \) of length \( \geq 2 \) and let \( v \) be an interior vertex. Unless for one of the end vertices of \( P \) both other incident multipaths are thin, \( G \) may be reconstructed from \( G_v \). Thus, there must be an E-vertex \( w \) with all 3 incident multipaths thin. If in the triangle of remaining multipaths two of them are fat then the E-vertices in \( G_v \) may be found and \( G \) reconstructed.

Suppose there is a fat multipath \( M \) and 5 thin multipaths, at least one of which has length \( \geq 2 \). If the multipath opposite \( M \) has length \( \geq 2 \) then \( G \) is reconstructible from \( G_w \), else \( G \) is reconstructible from \( G_v \), where \( v \) and \( w \) are as above.

Suppose all multipaths are thin. Let \( q \) be the number of multipaths which are edges; this is half the sum of the numbers at the E-vertices. If \( q \geq 5 \) there is a triangle of edges. If there is a triangle of edges let \( v \) be the E-vertex not on it; then \( G \) is reconstructible from \( G_v \).
Suppose it is known that there is a parallel pair of edges; $G$ is reconstructible from $G_v$ where $v$ is an interior vertex of a thin multipath of length $\geq 2$. If $3 \leq q \leq 4$ and there is no triangle then there is a parallel pair of edges.

Say that an E-vertex is connectible if the 3 incident thin multipaths have length $\geq 2$; if $v$ is a connectible E-vertex then $G$ is reconstructible from $G_v$. If $q \leq 1$ or $q = 2$ and the edges are adjacent then there is a connectible E-vertex.

For the rest of the section it will be assumed that $G$ has limbs. If $i, j$ are E-vertices call the L-multipath joining $i$ and $j$ the $ij$ multipath.

**Lemma 20.** If $G$ has more than 1 E-limb then $G$ is reconstructible.

**Proof.** Let $G^I$ be $G$, with the E-limbs removed; $G^I$ is known from $G_u$ where $u$ is a degree 1 vertex of an E-limb. Consider the orbits of the automorphism group of $G^I$.

By a typical argument, for each orbit the E-limbs attached to the nodes of the orbit may be determined. Let $r$ be the number of E-limbs of size 1; then either $r > 1$, or $r = 1$ and there are E-limbs of size $\geq 1$. The E-limbs of size $> 1$ belonging to each orbit may be determined from $G_u$ where $u$ is the degree 1 vertex of a size 1 E-limb. If $r > 1$ let $u$ range over the degree 1 vertices of size 1 E-limbs; the orbits containing size 1 E-limbs may be determined, and then the number in each such orbit, from the $G_u$. If $r = 1$ let $u$ be a degree 1 vertex of an E-limb of size 2; the orbit containing the size 1 E-limb may be determined from $G_u$.

The proof of the lemma is divided into cases, according to the orbit size list.

Case 1111. The vertex to which each E-limb is attached is known.

Case 112. The automorphism group acts transitively on the size 2 orbit, so the trees assigned to it may be assigned to the nodes of the orbit arbitrarily.

Case 22. Both orbits may be assumed to have a single E-limb, of size 1, since otherwise $G$ can be reconstructed from the $G_u$ where $u$ is a degree 1 vertex of a size 1 E-limb. If the automorphism group is $Z_2 \times Z_2$ (with generators (01),(23), say), the trees assigned to each orbit may be assigned to the nodes of the orbit arbitrarily.
Suppose the automorphism group is $Z_2$, w.l.g. with generator $(01)(23)$. The 01 L-multipath $A$ and the 23 L-multipath $A'$ are symmetric. Let $B$ denote the 02 and 13 L-multipaths and $C$ the 03 and 12. $B \neq C$, else the automorphism group is not $Z_2$.

If at least 1 of $B, C$ (w.l.g. $B$) is asymmetric let $u$ be a degree 1 vertex of an I-limb. In $G_u$ there is a unique copy of $B$, and it is readily determined whether the E-limbs are at the ends of a copy of $B$, and $G$ may be reconstructed from $G^I$. If $B, C$ are symmetric then $A \neq A'$. If at least 1 of $B, C$ (say $B$) has I-limbs let $u$ be a degree 1 vertex of an I-limb. The pair of $C$ L-multipaths may be found in $G_u$ and it may be determined whether the E-limbs are at the ends of a copy of $C$ and $G$ may be reconstructed from $G^I$. The argument if either $A$ or $A'$ has an I-limb is similar.

If at least one of $B, C$ is fat (say $B$) let $w$ be a vertex on a path of $B$ such that $G_w$ has only 2 limbs. The two parallel copies of $C$ may be found in $G_w$ and it may be determined whether the E-limbs are at the ends of a copy of $C$. The argument if either $A$ or $A'$ is fat is similar.

In the remaining case let $w$ be a vertex on the longer of $A, A'$ such that in $G_w$ there are two limbs. It may be determined from $G_w$ whether the limbs are at the ends of a copy of $B$ or $C$, and $G$ may be reconstructed from $G_w$.

Case 13. If the automorphism group is $S_3$ the 1-limbs on the size 3 orbit may be assigned arbitrarily to the nodes of the orbit. Otherwise, numbering so that the size 3 orbit is 012, there is an asymmetric L-multipath $A$ such that the 01, 12, and 20 L-multipaths all equal $A$. $A$ must have I-limbs, and $G$ is reconstructible from $G_u$ where $u$ is a degree 1 vertex of an I-limb if $A$.

Case 4. If the automorphism group is $Z_2 \times Z_2$ the 01,23 multipaths are both $A$, the 02,13 multipaths are both $B$, and the 03,12 multipaths are both $C$, where $A$, $B$, and $C$ are symmetric and all distinct. The proof is divided into subcases, according to the E-limb size list $S$. If $S$ is 11 the proof is similar to the previous case of orbit sizes 22 and automorphism group $Z_2 \times Z_2$. If there are any I-limbs let $u$ be a degree 1 vertex of one; then which of $A, B, C$ join the 2 E-limbs may be determined from $G_w$ and $G$ reconstructed from $G^I$. It there is a fat multipath let $w$ be a vertex on a path of one such that $G_w$ has only 2 limbs; which of $A, B, C$ joins them may be determined from $G_w$. In the remaining case let $w$ be a vertex
along the longest of $A, B, C$ such that $G_w$ has 2 limbs.

If $S$ is 12 the L-multipath joining the two E-limbs can be determined from $G_u$ where $u$ is a degree 1 vertex of the size 2 E-limb. If $S$ is 111 the E-limbs may be assigned to E-vertices arbitrarily. If $S$ is 112 the two L-multipaths joining the size 2 E-limb to a size 1 E-limb are determined from the $G_u$ where $u$ is the degree 1 vertex of a size 1 E-limb. If $S$ is 122 the L-multipath joining the 2 size 2 E-limbs can be determined from $G_u$ where $u$ is the degree 1 vertex of the size 1 E-limb. If $S$ is 123 the L-multipath joining the size 2 and size 3 E-limb may be determined from $G_u$ where $u$ is the degree 1 vertex of the size 1 E-limb; and the L-multipath joining the size 1 and size 2 E-limb may be determined from $G_u$ where $u$ is a degree 1 vertex of the size 2 E-limb.

If the automorphism group is $Z_4$ the nodes may be numbered so that the 01, 12, 23, and 30 L-multipaths are all equal to an asymmetric L-multipath $A$. $G$ is readily reconstructed from $G_u$ where $u$ is a degree 1 vertex of an I-limb of $A$.

If the automorphism group is $D_8$ the graph consists of 3 parallel classes of L-multipaths, two of which are equal. The argument is essentially the same as in the case of 3 distinct parallel classes given above.

In the remaining case the automorphism group is $S_4$ and the E-limbs may be assigned arbitrarily to the E-vertices.

**Lemma 21.** If $G$ has 1 E-limb and the sum of the sizes of the I-limbs is $\geq 2$ then $G$ is reconstructible.

**Proof.** Let $v$ denote the E-vertex with the E-limb attached. Let $\Sigma_1$ denote the set of L-multipaths incident to $v$, with $v$ marked. Let $\Sigma_2$ denote the remaining 3 L-multipaths. For $i = 1, 2$ let $s_i$ denote the sum of the sizes of the limbs of $\Sigma_i$; by a standard argument $s_1$ and $s_2$ may be determined, and then $\Sigma_1$ and $\Sigma_2$, from the $G_u$ where $u$ ranges over the degree 1 vertices of the I-limbs. By an argument as in theorem 10 the L-multipath sizes in both $\Sigma_1$ and $\Sigma_2$ must form an interval starting at 0. If $s_1 = 0$ then $v$ is the only E-vertex with 3 incident size 0 L-multipaths, else there is only 1 L-multipath of nonzero size and $G$ is reconstructible by theorem 10. Likewise, $s_2 \neq 0$ may be assumed.

Let $G^E$ be $G$ with I-limbs deleted. If $G^E$ is rigid $G$ may be reconstructed by taking the union of the $G_u$ where $u$ ranges over
degree 1 vertices of I-limbs. Otherwise, number E-vertices so that the E-vertex with the E-limb attached is \( v \) is 0, and (12) is an automorphism of \( G^E \).

Suppose the 03 multipath \( B \) differs from the 01 and 02 multipaths \( A \). If the 12 multipath has any I-limbs then \( G \) is reconstructible from \( G_u \) where \( u \) is a degree 1 vertex of one; likewise for the 03 L-multipath. If the 01 and 02 L-multipaths both have nonzero size then \( G \) is reconstructible from \( G_u \) where \( u \) is a degree 1 vertex of an I-limb on a copy of \( A \) of smallest size. If the 03 and 13 multipaths differ from \( B \), likewise for them. If they equal \( B \) there is only one vertex in \( G^I \) with 3 incident equal multipaths, and \( G \) is reconstructible from \( G^I \). Thus, there are 2 nonzero size L-multipaths. From \( G^I \) it is known whether they are vertex disjoint, and so \( G \) is reconstructible from \( G_u \) where \( u \) is the degree 1 vertex of the I-limb on a copy of \( A \).

Suppose \( B = A \). If there is only 1 E-vertex with 3 incident multipaths equaling \( A \) then \( G \) is reconstructible from \( G^I \). Otherwise, all multipaths are \( A \), except possibly the 02 multipath. If it differs, similarly to arguments just given, it must be size 0, whence the 02 multipath must be size 0, and the argument proceeds as before.

Suppose all 6 multipaths equal \( A \). If there are 2 nonzero size L-multipaths in \( \Sigma \), w.l.g. the 12 L-multipath may be assumed to have size 0. As usual the 03 L-multipath has size 0, and so only one L-multipath in \( \Sigma \) has nonzero size, say the 01 L-multipath. \( G \) may be reconstructed from \( G_u \) where \( u \) is a degree 1 vertex of an I-limb of the 13 L-multipath. In the remaining case of two nonzero size multipaths the argument is as before. \( \square \)

**Lemma 22.** If \( G \) has 1 E-limb and a single I-limb of size 1 then \( G \) is reconstructible.

**Proof.** Let \( v, \Sigma_1, \Sigma_2 \) be as in the previous proof. Let \( A \) be the L-multipath of nonzero size. \( A \) is in \( \Sigma_1 \) iff there is a vertex \( w \) on a path, such that \( G_w \) has only 1 limb.

Suppose \( A \) is in \( \Sigma_1 \). Let \( w \) be the root of the I-limb. From \( G_w \) the L-multipaths in \( \Sigma_1 \), with \( v \) marked, may be determined, and also the multipath joining the other ends of the two such other than \( A \). If \( \Sigma_1 \) contains only 1 copy of \( A \) then \( G \) may be reconstructed from \( G_u \) where \( u \) is the degree 1 vertex of the I-limb. Otherwise, let \( w \) be a vertex on a path of a second copy of \( A \) such that in \( G_w \) there
is only 2 limbs; \( v \) can be located in \( G_w \), and also the 4th E-vertex since \( B \) is known. Thus, \( G \) may be reconstructed from \( G_w \).

Lemma 23. If \( G \) has 1 E-limb and no I-limbs then \( G \) is reconstructible.

Proof. The argument in the proof of lemma 19 that \( G \) is reconstructible if there is a fat multipath carries through if there is a single E-limb, with some modifications.

Suppose the E-vertices are numbered so that 0 has 3 incident thin multipaths, and the 12 and 23 multipaths are fat. \( G \) is reconstructible from \( G_{v_0} \) unless the vertex with the E-limb attached is (w.l.g.) \( v_1 \) and the 03 multipath has length 2. In this case \( G \) is reconstructible from \( G_w \) where \( w \) is the vertex on the 03 path.

Suppose there is a fat multipath \( M \) and 5 thin multipaths, at least one of which has length \( \geq 2 \). The length \( l \) of the path opposite \( M \) is known since \( B \) is. If there is a path \( P \) of length \( \geq 2 \) incident to an E-vertex of \( M \) let \( w \) be a vertex of \( P \) such that in \( G_w \) there are as few limbs as possible. \( G \) is reconstructible from \( G_w \) since \( l \) is known. In the remaining case the paths incident to the E-vertices of \( M \) are edges, and the remaining path \( P \) has length \( \geq 2 \). \( G \) may be reconstructed from \( G_w \) where \( w \) is an interior vertex of \( P \) such that in \( G_w \) the number of limbs is as small as possible.

Suppose all multipaths are thin. Let \( v, \Sigma_1, \Sigma_2 \) be as in the previous proof. The multipaths in \( \Sigma_1 \) are all edges iff \( G_v \) has no limbs. If this is the case \( G \) is reconstructible from \( G_v \), since the lengths of all thin multipaths is known from \( G^l \).

Otherwise there is a vertex \( w \) on a path, such that \( G_w \) has a degree 2 vertex with a limb. From \( G_w \), \( \Sigma_1 \) may be determined. Let \( P_1, P_2, P_3 \) be the paths in nondecreasing order of length. Choosing \( w \) on \( P_3 \) the path \( Q_{13} \) between the ends of \( P_1 \) and \( P_2 \) is known. If \( P_2 \) has length 1 then \( G \) is reconstructible from \( G_w \). Otherwise choose \( w' \) on \( P_2 \) to obtain \( Q_{13} \); \( G \) may now be reconstructed.

Lemma 24. If \( G \) has a fat L-multipath then the L-multipaths are determined.

Proof. This is trivial if \( G \) has an E-limb; suppose not. By theorem 10 it suffices to consider the case of a single L-multipath \( M \) of nonzero size. If \( M \) is thin let \( v \) be a vertex on a path of a fat multipath. If \( M \) is fat let \( v \) be an E-vertex not incident to \( M \). In either case \( M \) is an L-multipath of \( G_v \).
Lemma 25. Suppose \( P \) is an L-path of length \( \geq 2 \) with its ends marked and \( v \) is a vertex on \( P \). Then either \( P \) lies over \( P_v \) in only one way, or \( P \) is symmetric.

Proof. Orient \( P \) and let \( T_1, \ldots, T_{l-1} \) be the trees along \( P \), where \( l \) is the length of \( P \). Say that \( P \) can proceed along \( P_v \) to \( i \) if it can proceed to \( i - 1 \) (vacuous if \( i = 1 \)), and a copy of \( T_i \) can be cut out of the remaining subtree of \( P \). It is readily verified that if \( P \) in both orientations can proceed along one of the two trees of \( P_v \) of maximum length to \( \lfloor (l - 1)/2 \rfloor \) then \( P \) is symmetric, proving the lemma.

Lemma 26. If \( G \) has no E-limbs and at least 1 fat L-multipath then \( G \) is reconstructible.

Proof. The proof is a modification of statements in the proof lemma 19. If for some \( m \geq 2 \) there is a multipath of multiplicity \( m \) but none of multiplicity \( m - 1 \), let \( v \) be a vertex on a path of such a multipath. Using lemmas 24 and 25, \( G \) is reconstructible from \( G_v \). Thus, the multipath multiplicities may be assumed to form an interval \([1, m]\).

Suppose there is a thin L-multipath \( P \) length \( \geq 2 \). Again using lemmas 24 and 25, \( G \) is reconstructible from \( G_v \) where \( v \) is on \( P \), unless there is an E-vertex \( v \) with 3 incident thin L-multipaths.

If \( G \) has 2 fat multipaths, using lemma 24 and subtracting the L-multipaths of \( G_v \), the multiset \( \Sigma \) of L-multipaths incident to \( v \) is known. If these all have size 0 \( G \) is reconstructible from \( G_v \). If \( \Sigma \) contains an L-multipath of length \( \geq 3 \) \( G \) is reconstructible from \( G_w \) where \( w \) is on such a path and at a distance \( \geq 2 \) from \( v \). Otherwise, \( G \) is reconstructible from \( G_v \).

Suppose there is a fat multipath \( M \) and 5 thin multipaths, at least one of which has length \( \geq 2 \). The argument is as in this case of the proof of lemma 23, except that \( w \) is arbitrary and lemmas 24 and 25 are used.

Thus, it may be assumed that all thin L-multipath are edges.

If there is an E-vertex \( v \) such that \( G_v \) is a triangle with some limbs attached, using lemma 24, \( G \) may be reconstructed from \( G_v \).

If the L-multipaths which are edges form a square let \( v \) be an interior vertex of a path on an L-multipath of multiplicity 2; then \( G \) is reconstructible from \( G_v \).
If the L-multipaths which are edges form a path of length 2 let \( v \) be an E-vertex such that \( G_v \) is a triangle of fat L-multipaths with one E-vertex having a limb. Then \( G \) is reconstructible from \( G_v \).

If the L-multipaths which are edges form a parallel pair let \( v \) be an interior vertex of a path on an L-multipath of multiplicity 2; then \( G \) is reconstructible from \( G_v \).

Suppose there is a single L-multipaths which is an edge, with E-vertices \( v_1, v_2, \) and \( v_3, v_4 \) the remaining E-vertices. By an argument in the proof of lemma 19, adapted using lemmas 24 and 25, every fat multipath must contain an edge. By considering \( G_v \) for \( v \in \{v_3, v_4\} \), the multiset \( \Sigma \) of L-multipaths incident to \( v_1 \) or \( v_2 \) may be determined. If an L-multipath of \( \Sigma \) has a path of length \( \geq 3 \), let \( M \) be one of least multiplicity among such, let \( P \) be a path of \( M \) of length \( \geq 3 \), and let \( w \) be the vertex of \( P \) such that in \( G_w \) there is only one limb, and it is attached to \( v_3 \) or \( v_4 \). \( M \) may be found in \( G_v \), and \( P \) may be determined, whence by lemma 25 \( G \) may be reconstructed.

If there is no such \( M \) in \( \Sigma \), let \( M \) be the L-multipath with ends \( v_3, v_4 \). If \( M \) has a path \( P \) of length \( \geq 3 \) proceed similarly. Otherwise, all paths which are not edges have length 2, and \( G \) is reconstructible from \( G_v \) where \( v \in \{v_3, v_4\} \).

\[ \text{Lemma 27. If } G \text{ has no E-limbs, all L-multipaths are thin, and there are } \geq 2 \text{ L-multipath of nonzero size, then } G \text{ is reconstructible.} \]

Proof. By theorem 10.b the L-paths are known. Let \( l_0 \) be the minimum length of a size 0 L-path. Let \( F \) denote the set of size 0 length \( l_0 \) paths.

If \( F \) contains a triangle, i.e., there is an E-vertex \( v \) such that in \( G_v \) there is a cycle of length \( 3l_0 \) with 3 limbs attached, \( G \) is reconstructible from \( G_v \).

If \( F \) is a square let \( u \) be the degree 1 vertex of the I-limb of an L-path of size 1; \( G \) is reconstructible from \( G_u \).

Suppose \( F \) is a path of length 2. Number the E-vertices so that along the path they are 102, and 3 is the remaining one. Let \( P \) be the 12 L-path; \( P \) is determined from any \( G_u \) where \( u \) is a degree 1 vertex of a limb not on \( P \). If \( P \) has size \( \geq 2 \) then \( G \) is reconstructible using lemma 4. Let \( \Sigma \) be the L-paths incident to vertex 3, with 3 marked. If \( P \) has size 1 \( \Sigma \) is known from \( G_u \) where \( u \) is the degree 1 vertex of the limb of \( P \); otherwise it may be determined by a
standard argument from the $G_u$ for $u$ a degree 1 vertex. As noted in earlier arguments, it may be assumed that the 03 L-path has size 0, and only one L-path in $\Sigma$ has nonzero size, say the 01 L-path. $G$ may be reconstructed from $G_u$ where $u$ is the degree 1 vertex of the I-limb of the 01 L-path.

Suppose $F$ is a parallel pair. If there is a path of length $\geq 3$ in $F^{cmpl}$, the complement of $F$, let $w$ be a vertex on such a path $P$ such that $G_w$ has as few limbs as possible. In $G_w$, 3 of the E-vertices of $G$ may be found from degree considerations, and the 4th using the fact that the two length $l_0$ L-paths are parallel; $G$ may now be reconstructed using lemma 25.

Suppose all L-paths in $F^{cmpl}$ have length 2. If $l_0 \geq 2$ there cannot be an L-path not in $F$ of size 0, whence $G$ is reconstructible from $G_u$ for $u$ the degree 1 vertex of the limb on an L-path of size 1. If $l_0 = 1$ there must be an L-path not in $F$ of size 0; let $w$ be its midpoint. $G$ is reconstructible from $G_w$, since on at least 1 path of length 3 the E-vertex may be located, and the other may then be by finding the parallel edge.

Suppose $F$ is a single L-path; let $\{0, 1\}$ be its incident E-vertices, and $\{2, 3\}$ the E-vertices incident to the path $P$ parallel to the edge. If $P$ has size $\geq 2$ then $G$ is reconstructible using lemma 4.

Suppose $P$ has size 1. If $P$ is symmetric then $G$ is reconstructible from $G_u$ where $u$ is the degree 1 vertex of the limb of $P$. Suppose $P$ is not symmetric. For $i \in \{2, 3\}$ let $Q_i$ be the path between 0 and 1 containing $i$, with attached limbs, and let $d_{ij}$ be the distance from $i$ to $j$, $j \in \{0, 1\}$. If $\{d_{20}, d_{21}\} \neq \{d_{30}, d_{31}\}$, let $u$ be the degree 1 vertex of the limb on $P$ and let $t$ be any other degree 1 vertex; the location of the limb on $P$ may be determined from $G_t$, and $G$ then reconstructed from $G_u$. Let $s_i$ be the size of $Q_i$, and assume $s_2 \leq s_3$; if $0 < s_2 < s_3$ the argument is similar. The remaining cases are as follows, noting that $s_2 \leq 2$ may be assumed.

01. The limb on $Q_3$ is between $j$ and 3 for exactly one $j \in \{0, 1\}$, and the distance from its root to $j$ is determined from $G_u$.

0, $\geq 2$. $G$ may be reconstructed using lemma 4 for $Q_3$.

11. Considering $Q_i$ as oriented, unless $Q_2 = Q_3$ or $Q_2 = Q_3^r$, the location of the limb on $P$ may be determined from the $G_i$. In the remaining cases the two possibilities are isomorphic.

22. Using lemma 4 it may be assumed that both $Q_i$ are sym-
metric. Unless \( Q_2 = Q_3 \) the location of the limb on \( P \) may be determined from the \( G_t \), and in the remaining case the two possibilities are isomorphic.

Suppose \( P \) has size 0, and let 0,1 etc. be as above. if \( P \) has length 2, which can occur only if \( t_0 = 1 \), then \( Q_2, Q_3 \) are known from \( G_w \) where \( w \) is the midpoint of \( P \). Unless \( s_2 = 1 \) and \( s_3 \leq 2 \) \( G \) is reconstructible by lemma 10.

In case 11 of \( s_2, s_3 \), unless \( Q_2 = Q_3 \) or \( Q_2 = Q_5^r \), \( G \) is reconstructible from either \( G_t \) using \( G_w \), and in the remaining cases the two possibilities are isomorphic.

In case 12, unless \( Q_3 \) is symmetric and \( Q_2 \) is not \( Q_3 \) \( t \) where \( t \) is a degree 1 vertex of a limb on \( Q_3 \), \( G \) is reconstructible from \( G_t \) using \( G_w \), where \( t \) is the degree 1 vertex of the limb on \( Q_2 \). Unless \( \{d_{20}, d_{21}\} = \{d_{30}, d_{31}\} \), \( G \) may be reconstructed from \( G_t \) for \( t \) the degree 1 vertex of a limb of \( Q_3 \). If \( d_{20} = d_{30} \) \( G \) is reconstructible from \( G_t \) where in \( G_t, Q_2 = Q_3 \) and if if \( d_{20} = d_{31} \) \( G \) is reconstructible from \( G_t \) where in \( G_t, Q_2 = Q_3^r \).

Suppose \( P \) has length \( \geq 3 \). If \( 2 \leq s_2 < s_3 \), \( Q_2 \) is known from \( G_w \) where \( w \) is a vertex on \( P \) such that in \( G_w \) \( Q_2 \) has size 2; \( G \) may be reconstructed using lemma 4. If \( 2 \leq s_2 = s_3 \), \( Q_2, Q_3 \) are known from the \( G_w \) where \( w \) is a vertex on \( P \) such that in \( G_w \) either \( Q_2 \) or \( Q_3 \) has size 2; \( G \) is reconstructible by lemma 10. If \( s_2 = 1 \) and \( s_3 \geq 3 \), \( Q_3 \) is known from \( G_t \) where \( t \) is the degree 1 vertex of the limb on \( Q_2 \); \( G \) may be reconstructed using lemma 10.

In case 11 of \( s_2, s_3 \), unless \( \{d_{20}, d_{21}\} = \{d_{30}, d_{31}\} \), \( G \) may be reconstructed from the \( G_t \). This is also true unless \( d_{20} = d_{21} \), and in that case \( G \) is reconstructible from \( G_w \) where \( w \) is a vertex along \( P \).

In case 12, again \( \{d_{20}, d_{21}\} = \{d_{30}, d_{31}\} \) may be assumed. Letting \( w \) be the vertex on \( P \) such that in \( G_w \) \( Q_3 \) has 3 limbs, and \( t \) the degree 1 vertex of the limb on \( Q_2 \), \( G_w \) lies over \( G_t \) in only one way, unless \( Q_3 \) is symmetric and \( v_3 \) is the midpoint. In that case, \( \{d_{20}, d_{21}\} \) is known from \( G + t \) where \( t \) is a degree 1 vertex of a limb on \( Q_3 \). \( G \) may then be reconstructed from \( G_w \) where \( w \) is a vertex along \( P \) such that in \( G_w \) \( Q_3 \) has 2 limbs.

Lemma 28. If \( G \) has no E-limbs, all L-multipaths are thin, and there is only 1 L-multipath of nonzero size, then \( G \) is reconstructible.

Proof. Let \( P \) be the path with limbs. Suppose there is a limb of
size $\geq 2$. If there is only one limb $G$ is reconstructible. Otherwise let $R$ be $P$ with the roots of limbs marked; $R$ is determined from $G_u$ where $u$ is the degree 1 vertex of a limb of size $\geq 2$. Let $t$ be the degree 1 vertex of a limb of size 1. Either $R$ fits over $P_t$ in only one way, or $R$ is symmetric; in either case $G$ is reconstructible from $G_t$.

Suppose $P$ has length $\geq 4$. Let $w$ be on $P$, such that in $G_w$ there are 2 limbs, one of size 1, or one of size 2 if there is no $w$ with one of size 1. $G$ is reconstructible from $G_w$.

If a size 0 L-path has length $\geq 6$ $G$ is reconstructible from $G_w$ where $w$ is at distance $\geq 3$ from both ends. There remain only a finite number of graphs. A computer program was written to verify that these all have distinct decks. The “Nauty” library [5] was used for graph canonicalization.

**Theorem 29.** If $G$ is an SBT graph with $B_m = K_4$ then $G$ is reconstructible.

**Proof.** This follows by lemmas 19 to 28.

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**Theorem 30.** If $n_p = 2$ then $G$ is reconstructible.

**Proof.** When $n_p = 2$ there are 4 S-minimal graphs; see figure 2 of [4]. From left to right the multipath graphs are an edge, a triangle, a square, and $K_4$. In the first case, $G$ is reconstructible by theorem 8. In the second and third cases $G$ is reconstructible by theorem 18. In the fourth case $G$ is reconstructible by theorem 29.

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**10. Remarks on $n_p = 3$**

Figure 1 shows the 17 S-minimal graphs for $n_p = 3$, each given as its multipath graph, with multipath multiplicities indicated. Numbering from 1, left to right, top to bottom, SBT graphs with $B_m$ equaling graph 1 are reconstructible by theorem 8, those with $B_m$
Figure 1: S-minimal graphs for $n_p = 3$
equaling graph 2-7 are reconstructible by theorem 18, and those with $B_m$ equaling graph 14 are reconstructible by theorem 29.

As noted earlier, any inseparable graph of size $\geq 2$ can occur as $B_m$. The simplest $B_m$ yet to be considered is that of graphs 10-11 of figure 1. It is clear, though, that even though the graph reconstruction conjecture seems likely to be true, the case-by-case methods of this paper are inadequate to make more substantial progress.

References

http://dx.doi.org/10.2140/pjm.1969.31.281

http://dx.doi.org/10.12732/ijpam.v87i4.2

http://dx.doi.org/10.12732/ijpam.v90i1.4

http://dx.doi.org/10.12732/ijpam.v95i2.14

http://dx.doi.org/10.1016/j.jsc.2013.09.003.