A QUESTION ON INDISCERNIBLES

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Abstract: The question is considered, whether for some limit ordinal \(\alpha\), \(L_\alpha\) has an infinite set of indiscernibles. This is true if \(\alpha\) is an \(\omega\)-Erdos cardinal. Whether the hypothesis can be weakened is a question of interest.

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1. Introduction

Let II denote the statement: for some limit ordinal \(\alpha\), \(L_\alpha\) has an infinite set of indiscernibles (ordinals equipped with their natural order). It is well-known that if there is an \(\omega\)-Erdos cardinal (a cardinal \(\kappa\) such that \(\kappa \rightarrow (\omega)^<\omega\)) then II holds (see theorem 9.3 of [2]). In particular \(\neg II\) is a very strong statement, implying that \(\omega\)-Erdos cardinals do not exist.

It is a question of interest whether II be deduced from a weaker hypothesis than the existence of an \(\omega\)-Erdos cardinal. It is also of interest what properties \(\alpha\) must have for \(L_\alpha\) to have indiscernibles.

It is also of interest whether \(II^L\) holds. Since \(\alpha \mapsto L_\alpha\) and the satisfaction predicate are absolute, \(II^L\) holds iff there as a limit ordinal \(\alpha\) and a set \(I \in L\) such that \(I\) is a set of indiscernibles for \(L_\alpha\).
Theorem 1. If $\Pi^L$ holds then $\Pi$ holds.

Proof. This follows by the remarks preceding the theorem. □

Since theorem 9.3 of [2] holds in $L$, $\Pi^L$ holds if there is an $\omega$-Erdos cardinal in $L$, and this holds if there is an $\omega$-Erdos cardinal (theorem 9.15 of [2]),

2. Basic facts

It is well-known (see [1]) that there is a collection of function definitions $\{h_\phi\}$ such that $h_\phi$ defines a Skolem function for $\phi$ in $L_\alpha$ for any limit ordinal $\alpha$. The function defined in $L_\alpha$ will be denoted $h_\phi^{L_\alpha}$, or $h_\phi$ if there is no danger of confusion. The Skolem hull of $S \subseteq L_\alpha$ will always be taken using these functions, and denoted $H(S)$.

Let $I$ be a set of indiscernibles for $L_\alpha$. For $S \subseteq L_\alpha$ the transitive collapse of $H(S)$ is isomorphic to $L_{\tilde{\alpha}}$ for some $\tilde{\alpha}$; the composition $j : L_{\tilde{\alpha}} \hookrightarrow L_\alpha$ of the isomorphism with inclusion is an elementary embedding. Consequently, $j^{-1}[S]$ is a set of indiscernibles for $L_{\tilde{\alpha}}$.

Theorem 2. If $\Pi$ holds then there is a countable $\alpha$ such that $L_\alpha$ has an infinite set of indiscernibles $I$, and such that $L_\alpha = H(I)$.

Proof. Let $J$ be a set of indiscernibles for $L_\beta$. Let $S$ be the first $\omega$ elements of $J$. Let $L_\alpha$ be the transitive collapse of $H(S)$. Let $I = j^{-1}[S]$. □

Theorem 3. If $\Pi$ holds then it is not provable in ZFC that $\Pi$ implies the existence of inaccessible cardinals.

Proof. By theorem 2 and absoluteness, if $\Pi$ holds then it holds in $V_\kappa$ where $\kappa$ is the smallest inaccessible. If it were provable that $\Pi$ implied that an inaccessible cardinal existed, then an inaccessible cardinal would exist in $V_\kappa$, which is a contradiction. □

Theorem 4. If $\Pi^L$ holds then there is an $\alpha < \omega^L_1$ such that $L_\alpha$ has an infinite set of indiscernibles $I \in L$, and such that $L_\alpha = H(I)$.

Proof. The proof of theorem 2 is an argument in ZFC. Note that by absoluteness $H(S)$ is the same in $L$ and $V$. □
Theorem 5. If $II^L$ holds then it is not provable in ZFC that $II^L$ implies the existence of inaccessible cardinals in $L$.

Proof. As in the proof of theorem 3, if $II^L$ holds then it holds in $L_\kappa$ where $\kappa$ is the smallest inaccessible in $L$. \qed

3. $F_n$-indiscernibles

Let $F$ be the class of augmented formulas in the language of set theory expanded by symbols for the Skolem functions, where an augmented formula $\phi(x_1, \ldots, x_n)$ is a formula $\phi$ together with a sequence $x_1, \ldots, x_n$ of variables, which includes the free variables of $\phi$. For $C \subseteq F$ and $\alpha$ a limit ordinal, a subset $I \subseteq \alpha$ is said to be a set of $C$-indiscernibles for $L_\alpha$ if for all $\phi(x_1, \ldots, x_n) \in C$, and sequences $\gamma_1 < \cdots < \gamma_n$ and $\delta_1 < \cdots < \delta_n$ of elements of $I$, $\models_{L_\alpha} \phi(\gamma_1, \ldots, \gamma_n) \iff \phi(\delta_1, \ldots, \delta_n)$. $F$-indiscernibles are called simply indiscernibles.

Let $F_n$ denote the formulas of $F$, where the variable sequence has length at most $n$. For a cardinal $\kappa$ and an integer $n$ let $IE(\kappa, n)$ be defined by the recursion: $IE(\kappa, 0) = \kappa$, $IE(\kappa, n + 1) = 2^{IE(\kappa, n)}$.

Theorem 6. For an integer $n > 0$, $L_\kappa$ has a set of $F_n$-indiscernibles of order type $(2^{\aleph_0})^+$ where $\kappa = IE(\aleph_0, n)^+$.

Proof. By the Erdos-Rado theorem (theorem 7.3 of [2]), $\kappa \rightarrow ((2^{\aleph_0})^+)^n_{\aleph_0}$. As in the proof of lemma 17.24 of [1], let $F : [\kappa]^n \mapsto$ Pow($F_n$) be the function where $F(\gamma_1, \ldots, \gamma_n) = \{\phi(x_1, \ldots, x_n) \in F_n : \models_{L_\kappa} \phi(\gamma_1, \ldots, \gamma_n)\}$. There is a homogeneous set for this partition, and it is a set of indiscernibles as required. \qed

4. Atomic formulas

Let $A$ be the set of atomic formulas of $F$, and let $A_n$ be the set of atomic formulas of $F_n$.

Theorem 7. A set of $A$-indiscernibles for $L_\alpha$ is a set of $F$-indiscernibles. A set of $A_n$-indiscernibles for $L_\alpha$ is a set of $F_n$-indiscernibles.
Proof. Let $I$ be a set of $A$-indiscernibles. By induction on the formation of $\phi$, $I$ is a set of indiscernibles for $\phi$. This follows by hypothesis for atomic formulas. The induction step for a propositional connective is straightforward. For $\phi = \exists y \psi(y, \bar{x})$, inductively $I$ is a set of indiscernibles for $\psi(h_\psi(\bar{x}), \bar{x})$, and hence for $\phi$.

Subsets of $A$ lead to questions of interest. In particular, let $E$ be the set of equations. It is of interest whether there is an $L_\alpha$ with an infinite set of $E$-indiscernibles, or whether the value of $\kappa$ in theorem 6 can be improved for $E_n$-indiscernibles.

Let $E_r$ be the equations $y = t(\bar{x})$, where in the variable sequence for this formula, $y$ can occur at any point in $\bar{x}$.

**Theorem 8.** $I$ is a set of $E_r$-indiscernibles for $L_\alpha$ iff every formula of $E_r$ has the value false at sequences from $I$. The same holds for $E_{rn}$ for $n \in \omega$.

**Proof.** Suppose $I$ is a set of $E_r$-indiscernibles. Let $x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n$ be the variable list for $y = t$. Let $\alpha_1 < \cdots < \alpha_n$ be elements of $I$. It may be assumed that $\alpha_{i+1}$ in not the successor of $\alpha_i$ in the enumeration of $I$; let $\beta$ be the successor. If $y = t$ is true then $\beta = \alpha_i$, a contradiction. Hence $y = t$ is false. The converse implication is trivial.

The same questions can be asked for $E_r$ as for $E$. Let $E_{rl}$ be the equations of $E_r$, where $y$ is at the end of the variable sequence.

**Theorem 9.** $L_{\aleph_1}$ has a set of $E_{rl}$-indiscernibles of order type $\aleph_1$.

**Proof.** Define the element $i_\beta$ of $I$ recursively as the least element which is not in the Skolem hull of $\{i_\gamma : \gamma < \beta\}$.

**References**
