A QUESTION ON INDISCERNIBLES

Martin Dowd

60 Mooring Ln.
Daly City, CA 94014, USA
e-mail: MartDowd@aol.com

Abstract: The question is considered, whether for some limit ordinal α, L_α has an infinite set of indiscernibles. This is true if α is an ω-Erdos cardinal. Whether the hypothesis can be weakened is a question of interest.

AMS Subj. Classification: 03e55
Key Words: indiscernibles

1. Introduction

Let II denote the statement: for some limit ordinal α, L_α has an infinite set of indiscernibles (ordinals equipped with their natural order). It is well-known that if there is an ω-Erdos cardinal (a cardinal κ such that $\kappa \to (\omega)^{<\omega}$) then II holds (see theorem 9.3 of [2]). In particular $\neg$II is a very strong statement, implying that ω-Erdos cardinals do not exist.

It is a question of interest whether II be deduced from a weaker hypothesis than the existence of an ω-Erdos cardinal. It is also of interest what properties α must have for L_α to have indiscernibles.

It is also of interest whether II^L holds. Since $\alpha \mapsto L_\alpha$ and the satisfaction predicate are absolute, II^L holds iff there as a limit ordinal α and a set $I \in L$ such that I is a set of indiscernibles for L_α.

Theorem 1. *If II^L holds then II holds.*

Proof. This follows by the remarks preceding the theorem. □

Since theorem 9.3 of [2] holds in L, II^L holds if there is an ω-Erdos cardinal in L, and this holds if there is an ω-Erdos cardinal (theorem 9.15 of [2]),

2. Basic facts

It is well-known (see [1]) that there is a collection of function definitions $\{h_\phi\}$ such that h_ϕ defines a Skolem function for ϕ in L_α for any limit ordinal α. The function defined in L_α will be denoted $h_\phi^{L_\alpha}$, or h_ϕ if there is no danger of confusion. The Skolem hull of $S \subseteq L_\alpha$ will always be taken using these functions, and denoted $H(S)$.

Let I be a set of indiscernibles for L_α. For $S \subseteq L_\alpha$ the transitive collapse of $H(S)$ is isomorphic to $L_{\tilde{\alpha}}$ for some $\tilde{\alpha}$; the composition $j : L_{\tilde{\alpha}} \mapsto L_\alpha$ of the isomorphism with inclusion is an elementary embedding. Consequently, $j^{-1}[S]$ is a set of indiscernibles for $L_{\tilde{\alpha}}$.

Theorem 2. *If II holds then there is a countable α such that L_α has an infinite set of indiscernibles I, and such that $L_\alpha = H(I)$.*

Proof. Let J be a set of indiscernibles for L_β. Let S be the first ω elements of J. Let L_α be the transitive collapse of $H(S)$. Let $I = j^{-1}[S]$. □

Theorem 3. *If II holds then it is not provable in ZFC that II implies the existence of inaccessible cardinals.*

Proof. By theorem 2 and absoluteness, if II holds then it holds in V_κ where κ is the smallest inaccessible. If it were provable that II implied that an inaccessible cardinal existed, then an inaccessible cardinal would exist in V_κ, which is a contradiction. □

Theorem 4. *If II^L holds then there is an $\alpha < \omega_1^L$ such that L_α has an infinite set of indiscernibles $I \in L$, and such that $L_\alpha = H(I)$.*

Proof. The proof of theorem 2 is an argument in ZFC. Note that by absoluteness $H(S)$ is the same in L and V. □

Theorem 5. *If II^L holds then it is not provable in ZFC that II^L implies the existence of inaccessible cardinals in L.*

Proof. As in the proof of theorem 3, if II^L holds then it holds in L_κ where κ is the smallest inaccessible in L. □

3. F_n-indiscernibles

Let F be the class of augmented formulas in the language of set theory expanded by symbols for the Skolem functions, where an augmented formula $\phi(x_1, \ldots, x_n)$ is a formula ϕ together with a sequence $x_1, \ldots, x_n$ of variables, which includes the free variables of ϕ. For $C \subseteq F$ and α a limit ordinal, a subset $I \subseteq \alpha$ is said to be a set of C-indiscernibles for L_α if for all $\phi(x_1, \ldots, x_n) \in C$, and sequences $\gamma_1 < \cdots \gamma_n$ and $\delta_1 < \cdots < \delta_n$ of elements of I, $\models_{L_\alpha}$ $\phi(\gamma_1, \ldots, \gamma_n) \Leftrightarrow \phi(\delta_1, \ldots, \delta_n)$. F-indiscernibles are called simply indiscernibles.

Let F_n denote the formulas of F, where the variable sequence has length at most n. For a cardinal κ and an integer n let $\mathrm{IE}(\kappa, n)$ be defined by the recursion: $\mathrm{IE}(\kappa, 0) = \kappa$, $\mathrm{IE}(\kappa, n+1) = 2^{\mathrm{IE}(\kappa,n)}$.

Theorem 6. *For an integer $n > 0$, L_κ has a set of F_n-indiscernibles of order type $(2^{\aleph_0})^+$ where $\kappa = \mathrm{IE}(\aleph_0, n)^+$.*

Proof. By the Erdos-Rado theorem (theorem 7.3 of [2]), $\kappa \to ((2^{\aleph_0})^+)^n_{2^{\aleph_0}}$. As in the proof of lemma 17.24 of [1], let $F : [\kappa]^n \mapsto \mathrm{Pow}(F_n)$ be the function where $F(\gamma_1, \ldots, \gamma_n) = \{\phi(x_1, \ldots, x_n) \in F_n : \models_{L_\kappa} \phi(\gamma_1, \ldots, \gamma_n)\}$. There is a homogeneous set for this partition, and it is a set of indiscernibles as required. □

4. Atomic formulas

Let A be the set of atomic formulas of F, and let A_n be the set of atomic formulas of F_n.

Theorem 7. *A set of A-indiscernibles for L_α is a set of F-indiscernibles. A set of A_n-indiscernibles for L_α is a set of F_n-indiscernibles.*

Proof. Let I be a set of A-indiscernibles. By induction on the formation of ϕ, I is a set of indiscernibles for ϕ. This follows by hypothesis for atomic formulas. The induction step for a propositional connective is straightforward. For $\phi = \exists y \psi(y, \vec{x})$, inductively I is a set of indiscernibles for $\psi(h_\psi(\vec{x}), \vec{x})$, and hence for ϕ. $\square$

Subsets of A lead to questions of interest. In particular, let E be the set of equations. It is of interest whether there is an L_α with an infinite set of E-indiscernibles, or whether the value of κ in theorem 6 can be improved for E_n-indiscernibles.

Let E_r be the equations $y = t(\vec{x})$, where in the variable sequence for this formula, y can occur at any point in $\vec{x}$.

Theorem 8. *I is a set of E_r-indiscernibles for L_α iff every formula of E_r has the value false at sequences from I. The same holds for E_{rn} for $n \in \omega$.*

Proof. Suppose I is a set of E_r-indiscernibles. Let $x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n$ be the variable list for $y = t$. Let $\alpha_1 < \cdots < \alpha_n$ be elements of I. It may be assumed that α_{i+1} in not the successor of α_i in the enumeration of I; let β be the successor. If $y = t$ is true then $\beta = \alpha_i$, a contradiction. Hence $y = t$ is false. The converse implication is trivial. $\square$

The same questions can be asked for E_r as for E. Let E_{rl} be the equations of E_r, where y is at the end of the variable sequence.

Theorem 9. *$L_{\aleph_1}$ has a set of E_{rl}-indiscernibles of order type $\aleph_1$.*

Proof. Define the element i_β of I recursively as the least element which is not in the Skolem hull of $\{i_\gamma : \gamma < \beta\}$. $\square$

References

[1] T. Jech, *Set Theory*, Springer, Germany (2003).

[2] A. Kanamori, *The Higher Infinite*, Springer-Verlag, Germany (2003).