Remarks on Levy's Reflection Axiom

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Abstract. Adding higher types to set theory differs from adding inaccessible cardinals, in that higher type arguments apply to all sets rather than just ordinary ones. Levy's reflection axiom is justified, by considering the principle that we can pretend that the universe is a set, together with methods of [Ga67]. We reprove some results of [Ga67], and some facts about Levy's reflection axiom, including the fact that adding higher types yields no new theorems about sets. Some remarks on standard models are made. An obvious strengthening of Levy's axiom to higher types is considered, which implies the existence of indescribable cardinals. Other remarks about larger cardinals are made; some questions of [Gl73] are settled. Finally we argue that the evidence for $V{=}L$ is strong, and that CH is certainly true.

1. Introduction.

Early discussions of Russell's paradox attempted to distinguish classes from sets. As early as [Ca99] a distinction is drawn between classes and sets. In [Ru05] the position is taken that classes exist in "intension" rather than "extension". A class is a predicate on the universe of sets, which are the first order objects; the universe can be quantified over, truth can be defined by an intensional induction, and so forth, but classes cannot for example be considered as members of a collection.

On the other hand, a type structure can be erected on top of the sets, so that collections of classes are allowed. Extending this construction leads to the notion of an inaccessible cardinal. The first inaccessible cardinal represents a stage in the cumulative hierarchy where what used to be considered the universe of discourse is collected. To quote [Dr74], "talking about all classes is tantamount to saying that we have not taken all levels, with no end".

It is common mathematical practice to admit an inaccessible cardinal to bring higher type objects into the universe, a classic example being the category of categories and functors. Aside from this occasional utility, the existence of inaccessible cardinals is irrelevant to most of mathematics. Foundationally, it appears that inaccessible cardinals do exist. Admitting their existence blurs the distinction between sets and classes, raising new questions about the sense in which classes exist. If one admits that the cumulative hierarchy can always be extended to a next higher stage that satisfies second order replacement, then classes do not exist as collections; equivalently the universe of discourse of set theory cannot be collected.

One can make statements about all sets. A more accurate view than Russell's position might be that the fact that the universe cannot be collected represents a limitation of first order logic. It should not therefore be surprising that adding higher types yields a formal system which still proves only true facts about sets. The semantics of this extension are what is vague. These considerations can be seen as a partial answer to Kreisel's [Kr67] question, "what are the proper laws (the 'logic') satisifed by the intensional element of the crude mixture". A deeper question is whether one can add any axioms other than those which arise only from logical considerations.

The foregoing can also be seen as justifying the existence of various large cardinals, a point which various authors have observed. For example, there is no largest inaccessible cardinal, because if there were the universe containing it could be collected (this axiom is called Tarski's axiom by some authors). The smallest universe containing an unbounded sequence of inaccessibles gives rise to a 1-inaccessible, where we define κ to be 0-inaccessible if it is inaccessible; $\alpha+1$-inaccesible if there are κ α-inaccessibles below it; and α-inaccessible for limit α if it is β-inaccessible for $\beta<\alpha$.

Granting α-inaccessibles for all α, we again collect the universe and obtain an inaccessible κ which is κ-inaccessible. This is the least member of the "diagonal intersection" of the classes of α-inaccessibles, with a natural cofinal sequence, where the diagonal intersection $\triangle_\gamma X_\gamma$ of the sequence of classes $\langle X_\gamma\rangle$ equals $\{\gamma:\gamma\in X_\gamma\}$. Natural cofinal sequences can be obtained for inaccessibles of still higher type using the methods of [Ve08].

Before giving these, we introduce some notation which is used throughout the paper. By inaccessible or Mahlo we mean strongly, and say weakly explicitly.

- Ord for the class of ordinals, with $\alpha, \beta, \gamma, \delta$ denoting ordinals and ω the integers.
- Lim for the limit ordinals.
- Card for the class of cardinals, with κ, λ denoting cardinals.
- Reg for the regular cardinals, Regu for the regular uncountable cardinals.
- Inac for the inaccessible cardinals.
- $|X|$ for the cardinality of the set X.
- $\mathrm{Ot}(X)$ for the order type of the set X of ordinals, in natural order.
- $\mathrm{Cf}(\alpha)$ for the cofinality of α.
- $\mathrm{Fix}(X)$ for $X \cap \{\alpha \in \mathrm{Lim} : \mathrm{Ot}(X \cap \alpha) = \alpha\}$.
- V_α for stage α of the cumulative hierarchy.
- $M \prec N$ for, M is an elementary substructure of N, in the language of set theory.

Let Φ be the family of functions $\phi : \mathrm{Ord} \mapsto \mathrm{Ord}$ which are nonzero only finitely often. For $\phi \in \Phi$ let $m(\phi)$ denote $\max\{\alpha : \phi(\alpha) > 0\}$. For $\phi, \psi \in \Phi$ define $\phi < \psi$ to hold if $m(\phi) < m(\psi)$, or $m(\phi) = m(\psi)$ and there is a $\beta \le \alpha$ such that $\phi(\beta) < \psi(\beta)$ and $\phi(\gamma) = \psi(\gamma)$ for $\beta < \gamma \le \alpha$. For each $\phi \in \Phi$ a class $I(\phi)$ of inaccessibles may be defined. Using obvious notation,

$I() = \mathrm{Inac};$

$I(\alpha+1, \vec{\theta}) = \mathrm{Fix}(I(\alpha, \vec{\theta}));$

$I(\alpha, \vec{\theta}) = \bigcap_{\beta<\alpha} I(\beta, \vec{\theta}),\ \alpha \in \mathrm{Lim};$

$I(\vec{0}, \alpha+1_{\gamma+1}, \vec{\theta}) = \triangle_\gamma I(\vec{0}, \xi_\gamma, \alpha_{\gamma+1}, \vec{\theta});$

$I(\vec{0}, \alpha_{\gamma+1}, \vec{\theta}) = \bigcap_{\beta<\alpha} I(\vec{0}, \beta_{\gamma+1}, \vec{\theta}),\ \alpha \in \mathrm{Lim};$

$I(\vec{0}, \alpha+1_\gamma, \vec{\theta}) = \bigcap_{\gamma<\delta} I(\vec{0}, 1_\delta, \vec{0}, \alpha_\gamma, \vec{\theta}),\ \gamma \in \mathrm{Lim};$

$I(\vec{0}, \alpha_\gamma, \vec{\theta}) = \bigcap_{\beta<\alpha} I(\vec{0}, \beta_\gamma, \vec{\theta}),\ \alpha, \gamma \in \mathrm{Lim};$

Let $i(\alpha;\phi)$ denote the α-th inaccessible in $I(\phi)$, with $i(\alpha)$ for $i(\alpha;)$.

This process can obviously be continued. In [Ga67] it is shown that if any sequence of operations as above is allowed, and the universe is collected, the result is a Mahlo cardinal. Levy's reflection axiom [Le60] is

$$\exists\kappa \in \mathrm{Reg}(\forall\vec{x} \in V_\kappa(F \Leftrightarrow F^{V_\kappa})),$$

where F is a formula whose free variables are from $\vec{x}$. Let us call this scheme axiom R; it is equivalent to an axiom scheme, which we call axiom F (as in [Dr74]). Axiom F states that if F is a formula defining a closed and unbounded (in Ord) class of ordinals then F is satisfied by a regular cardinal.

A simplified version of Gaifman's theory is given in section 4, and some additional remarks on Mahlo cardinals made. It is fair to say that the existence of Mahlo cardinals has been completely justified in terms of collecting the universe. It seems that by the same token axiom R is justified. The universe should be more inaccessible than any cardinal it contains, so if Mahlo cardinals exist axiom R surely holds. This may be seen as a consequence of the principle that even though the universe cannot be collected, we can always pretend that it can be.

Section 2 gives a discussion of weakened versions of axiom R. Section 3 gives a formal system containing higher types, and it is shown that adding higher types to ZFC+R yields a conservative extension. In section 5 it is observed that in ZFC+R+"higher types" it follows that the set of $\kappa \in \mathrm{Reg}$ for which $V_\kappa \prec V$ form an unbounded class, which is closed in Reg; other remarks on standard models are also made. In section 6 we discuss a strengthening of Levy's axiom which implies the existence of indescribable cardinals. Some remarks are made also about Π^1_1-indescribable cardinals; in particular, it is shown that the greatly Mahlo cardinals are stationary below one, a question left open by [Gl73]. Finally in section 7 we argue that the possible truth of the hypothesis of constructibility should be taken more seriously.

2. Weak versions of axiom R.

Axiom R is equivalent to variations where the existence of arbitrarily large inaccessibles is asserted, or where several formulas are reflected (see [Dr74]). It may also be weakened, for example to reflect a sentance with parameters, namely,

$$F \Rightarrow \exists\kappa \in \mathrm{Reg}(\vec{x} \in V_\kappa \wedge F^{V_\kappa}),$$

where the free variables of the formula F are from $\vec{x}$. This is easily seen (since $F \vee \neg F$) to be equivalent to

$$\exists\kappa \in \mathrm{Reg}(\vec{x} \in V_\kappa \wedge (F \Leftrightarrow F^{V_\kappa})),$$

where the free variables of F are from $\vec{x}$ and κ is not in $\vec{x}$, and to the variations for arbitrarily large κ or several formulas. It is an example of what [Gl73] calls partial reflection.

Let us call this axiom W. If F is restricted to be a sentance, call it axiom WS, and if in addition arbitrarily large κ are asserted to exist, axiom WSU. Clearly R⇒W⇒WSU⇒WS. We will show in section 4 that W⇒R. From the next two theorems, and results of section 4, these are the only implications which hold.

Theorem 1. From WS it follows that $i(n)$ exists for each $n \in \omega$; however $\forall n(i(n)$ exists) does not follow. If α is least such that $V_{i(\alpha)}$ is a model of WS then α is countable. If there are ω inaccessibles then WS is consistent.

Proof: Let G_n be the statement, "there are $\geq n$ inaccessibles". From WS we have $G_n \Rightarrow G_{n+1}$ and the first claim follows. For the second claim, by compactness it suffices to show that for each $F_1, \ldots, F_m$, there is a k such that $V_{i(k)}$ is a model of the instance of WS for each F_l. Consider the structures $V_{i(j)}$, $0 \leq j \leq 2^m$. Two of these must yield the same truth value for the F_l, and the larger j yields the required model. For the third claim, for each $\alpha' < \alpha$ there is a sentance $F_{\alpha'}$ which is true in $V_{i(\alpha')}$ but false in $V_{i(\alpha'')}$ for $\alpha'' < \alpha'$; the $F_{\alpha'}$ are distinct and the claim follows. The last claim follows since then every finite set of instances has a model.

THEOREM 2. From WSU it follows that $I(n)$ is unbounded for each $n \in \omega$; however $\forall n(I(n)$ is unbounded) does not follow.

PROOF: Proceeding as in theorem 1, for the first claim, let G_0 be "true" and G_{n+1} "$I(n)$ is unbounded". For the second claim we proceed by induction on m to find a $V_{i(\alpha;n+1)}$. For the basis, if the instance of WSU is false at each $i(l;n+1)$ it is true at $i(\omega;n+1)$. The induction step is similar and left to the reader.

3. Axiom R and type theory.

Attempts were made very early in set theory [Ru08] to define the sets using the type hierarchy, and there is still some interest in the subject [An65], even though the cumulative hierarchy has proved much more convenient. Adding higher types to the first order theory of Zermelo and Fraenkel has been considered, Bernays-Morse set theory being an example. Adding higher types has proved quite useful over the integers [Fe77]. Over the sets, the ordinals can be used as type indices; we give such a system here.

We use ZFCT to denote the system; the language of set theory is expanded with a unary predicate M and a unary function T. The axioms include:

- extensionality;
- the remaining axioms of ZFC relativized to M;
- $M(x)$ and $y \in x$ imply $M(y)$;
- $T(x)$ is an ordinal (i.e., an ordinal set);
- $T(x) = 0$ iff $M(x)$;
- if $T(x) > 0$ then $T(x) = \sup\{T(y) + 1 : y \in x\}$.

We use $x^{\leq\alpha}$ to denote a variable relativized to $T(x) \leq \alpha$, and similarly for $x^{<\alpha}$. The comprehension axiom scheme is:

- $\forall\alpha > 0\forall\vec{p}^{\leq\alpha}\exists x^{\leq\alpha}\forall w^{<\alpha}(w \in x \Leftrightarrow F)$,

where F is a formula whose free variables are among $w, \vec{p}$. The restriction on $\vec{p}$ is for convenience below.

The remaining two axioms, replacement and choice, require a "flat" pairing function, which codes n-tuples of objects of type $\alpha > 0$ as objects of type α. A function which works in our context can be found in [Dr74], where it is attributed to Quine. The n-tuple $\langle x_0, \ldots, x_{n-1}\rangle$ is coded as $\{f_{ni}(w) : w \in x_i, i < n\}$, where $f_{ni}(w) = (w - \omega) \cup \{i\} \cup \{n + k : k \in w \cap \omega\}$. Relations and functions may now be coded, as sets of codes of n-tuples. The usual notation for relations may be used; note that the arguments of a relation have lower type than the relation.

One version of higher type choice,

- $\forall\alpha > 0 \exists f^{\leq\alpha}(\forall x^{<\alpha}\exists! w f(w,x) \wedge \forall x \neq \emptyset \forall w(f(w,x) \Rightarrow w \in x))$,

states that there is a global choice function for each level. Also necessary is second order replacement,

- $\forall f^{\leq 1}\forall \vec{p}^0(\forall u^0 \exists! w^0 f(w,u,\vec{p}) \Rightarrow \forall u^0 \exists y^0 \forall w^0(w \in y \Leftrightarrow \exists u \in x f(w,u,\vec{p})))$.

There are various redundancies in these axioms; however it is convenient to include them all, especially when restricting the system to objects of type $< \beta$ where $\beta > 0$ is a definable ordinal. For such β, the system ZFCT_β adds the axioms $T(x) < \beta$; restricts α in comprehension and higher type choice to be $< \beta$; and relativizes F in comprehension to type $< \beta$. Note that comprehension, higher type choice, and second order replacement become vacuous if $\beta = 1$, so that ZFCT_1 is just ZFC. ZFCT_2, or second order set theory, is often called Bernays-Morse set theory. Gödel-Bernays set theory is obtained from ZFCT_2 by restricting F and $\vec{p}$ in comprehension to type 0. The system ZFCT_ω is mentioned in [Co66], and it is observed that its consistency follows from the existence of an inaccessible cardinal; indeed, the consistency of ZFCT follows. Another standard consistency result states that the consistency of ZFCT_β follows in $\mathrm{ZFCT}_{\beta+1}$; we will outline a proof.

The first step is to define $\mathrm{Tru}^\beta(f,x)$, which is the truth value of the type $< \beta$ formula f at the assignment $x = \langle x_0, \ldots, x_n \rangle$ to the free variables in alphabetic order. This convention complicates the recursive definition somewhat, but results in a predicate which is easy to use. Let TDF denote the recursion equations for Tru; as usual we will not give these explicitly since they are obvious.

An object of type β satisfying TDF can be defined in a well known manner, and the equations of TDF proved. For some details, a subset of a set is a set; it follows that inductions over ω on formulas of any type are valid. A sequence $\langle X_i : i < \omega \rangle$ of objects of type $\leq \beta$ can be coded as a single object of type $\leq \beta$, similar to the coding of n-tuples. To define Tru, the sequence $\langle \{x : \mathrm{Tru}(g,x)\} : g \leq f \rangle$ is defined. The scheme

- $\mathrm{Tru}(\ulcorner F \urcorner, x) \Leftrightarrow F$, where x denotes the appropriate function of the free variables of F,

follows readily from TDF. Also from TDF it follows that the axioms and rules of logic are valid, where now formulas are quantified over.

THEOREM 3. The consistency of ZFCT_β is provable in $\mathrm{ZFCT}_{\beta+1}$.

PROOF: From the above, if we can prove that the axioms are true, where they are quantified over, then it follows that any provable formula is true. The truth of single formula axioms is immediate, which leaves only the comprehension and first order replacement schemes. For comprehension,

$$\forall\alpha, 0 < \alpha < \beta, \forall f \in \omega \forall p^{\leq\alpha} \exists x^{\leq\alpha} \forall w^{<\alpha}(w \in x \Leftrightarrow \mathrm{Tru}(f, \langle w, p \rangle))$$

(where again $\langle w, p \rangle$ is an abbreviation) is a consequence of comprehension in $\mathrm{ZFCT}_{\beta+1}$, and verifies comprehension in ZFCT_β. The first order replacement scheme is similarly verified using second order replacement.

Theorem 4. ZFCT+R^M is a conservative extension of ZFC+R. The same is true if R is replaced by any of W, WS, WSU.

Proof: Let F be a first order sentance provable in ZFCT+R^M. It is readily verified that $\forall\kappa \in \text{Inac}(G \Rightarrow F)^{V_\kappa}$ is provable in ZFC, where G is the conjunction of the instances of R used in the ZFCT proof of F. Hence, $G \Rightarrow F$ is provable in ZFC+R, so F is. The proof for the other axioms is identical.

4. Systems of operations.

For $X \subseteq \text{Ord}$, define $\text{Lim}(X) = \{\alpha \in \text{Lim} : X \cap \alpha \text{ is unbounded in } \alpha\}$. This operation is monotone ($X \subseteq Y \Rightarrow \text{Lim}(X) \subseteq \text{Lim}(Y)$), and "local" ($\text{Lim}(X \cap \alpha) = \text{Lim}(X) \cap \alpha$). Also, $\text{Lim}(\text{Lim}(X)) \subseteq \text{Lim}(X)$ and $\text{Lim}(X \cup Y) = \text{Lim}(X) \cup \text{Lim}(Y)$.

Define $\text{Cl}(X) = X \cup \text{Lim}(X)$; $\text{Cl}(X)$ satisfies Kuratowski's closure axioms $\text{Cl}(\emptyset) = \emptyset$, $X \subseteq \text{Cl}(X)$, $\text{Cl}(\text{Cl}(X)) = \text{Cl}(X)$, and $\text{Cl}(X \cup Y) = \text{Cl}(X) \cup \text{Cl}(Y)$. It defines the Scott topology on Ord. The closed sets are those for which $\text{Lim}(X) \subseteq X$; for $X \subseteq Y$, X is closed in Y iff $Y \cap \text{Lim}(X) \subseteq X$. The intersection of any family of subsets of Y closed in Y is again closed in Y. We use the phrase "X closed in Y", implying that $X \subseteq Y$.

The operator $\text{Fix}(X)$ defined earlier is monotone and local, and $\text{Fix}(X) \subseteq X$. One easily checks that $\{\alpha \in \text{Lim} : \text{Ot}(X \cap \alpha) = \alpha\}$ is closed, so if X is closed then $\text{Fix}(X)$ is. If $\kappa \in X \cap \text{Card}$ then $\kappa \in \text{Fix}(X)$ iff $|X \cap \kappa| = \kappa$. If $\kappa \in X \cap \text{Reg}$ then $\kappa \in \text{Fix}(X)$ iff $\kappa \in \text{Lim}(X)$; in particular, if $X \subseteq \text{Reg}$ then $\text{Fix}(X) = X \cap \text{Lim}(X)$. If $D \subseteq \text{Reg}$ and X is closed in D then $\text{Fix}(X) = \text{Lim}(X) \cap D = \text{Fix}(\text{Cl}(X)) \cap D$.

If $\langle X_\gamma \rangle$ is a sequence of classes of ordinals, define the diagonal intersection $\triangle_\gamma X_\gamma$ to be $\{\alpha : \alpha \in X_\gamma \text{ for } \gamma \leq \alpha\}$. Note that $\triangle_\gamma X_\gamma \subseteq \bigcup_\gamma X_\gamma$. The operation is local; indeed $\triangle_\gamma (X_\gamma \cap \alpha) = (\triangle_{\gamma<\alpha} X_\gamma) \cap \alpha$. It is easily verified that if each X_γ is closed in Y, and $\langle X_\gamma \rangle$ is continuous (if $\gamma \in \text{Lim}$ then $X_\gamma = \bigcap_{\delta<\gamma} X_\delta$), then $\triangle_\gamma X_\gamma$ is closed in T. If $\langle X_\gamma \rangle$ is nonincreasing (if $\delta < \gamma$ then $X_\delta \subseteq X_\gamma$) then $\alpha \in \triangle_\gamma X_\gamma$ iff $\alpha \in X_\alpha$.

Some authors require only $\alpha \in X_\gamma$ for $\alpha < \gamma$; we write $\triangle^<$ for this operation. It has the advantage that continuity of a sequence of closed classes is not required for closure of the diagonal intersection, and the disadvantage that 0 is always in the diagonal intersection. The choice is one of convenience. We call a filter of subsets of κ, $\kappa \in \text{Regu}$, normal if it is closed under diagonal intersection of continuous sequences; a κ-complete filter is normal iff it is closed under $\triangle^<$ (provided it contains a set not containing 0).

For $D \subseteq \text{Lim}$, $\kappa \in \text{Regu}$, call X a D-club below κ if $X \cap \kappa$ is closed in $D \cap \kappa$ and unbounded below κ. We frequently omit the phrase "below κ", implying $X \subseteq \kappa$. If $D = \text{Lim}$, X is simply a club. We also consider clubs in ordinals $\alpha \in \text{Lim}$ with $\text{Cf}(\alpha) > \omega$. A subset $X \subseteq \alpha$ is called stationary if it has nonempty intersection with any club. Define $\text{Hu}(X)$ to be $\{\alpha \in \text{Lim} : \text{Cf}(\alpha) > \omega, X \cap \alpha \text{ stationary in } \alpha\}$, and $\text{H}(X) = X \cap \text{Hu}(X)$. H is monotone and local, and $\text{H}(X) \subseteq X$. The requirement $\mathrm{Cf}(\alpha) > \omega$ in the definition of $\mathrm{Hu}(X)$ is sometimes omitted; this adds to $\mathrm{Hu}(X)$ those α for which $\mathrm{Cf}(\alpha) = \omega$ and $X \cap \alpha$ is co-bounded in α.

LEMMA 5. If $D \subseteq \mathrm{Reg}$ and $\kappa \in \mathrm{H}(D)$ then

a. if X is a D-club then $\mathrm{Fix}(X)$ is a D-club;

b. if $\alpha < \kappa$ and X_γ is a D-club for $\gamma < \alpha$ then $\bigcap_{\gamma<\alpha} X_\gamma$ is a D-club; and

c. if X_γ is a D-club for $\gamma < \kappa$ and $\langle X_\gamma \rangle$ is continuous then $\triangle_\gamma X_\gamma$ is a D-club.

PROOF: For part a, $D \cap \kappa$ is stationary, and $\mathrm{Cl}(X)$ is a club, so $D \cap \mathrm{Fix}(\mathrm{Cl}(X))$ is unbounded; but this is $\mathrm{Fix}(X)$. For part b, similarly $D \cap \bigcap_{\gamma<\alpha} \mathrm{Cl}(X_\gamma)$ is unbounded; further this is contained in $\bigcap_{\gamma<\alpha} X_\gamma$. For part c, again $D \cap \triangle_\gamma \mathrm{Cl}(X_\gamma)$ is unbounded, and is contained in $\triangle_\gamma X_\gamma$.

For any $\kappa \in D$ the smallest family of sets containing $D \cap \kappa$ and closed under the operations of the lemma consists of those sets obtained from sequences of classes, of length $< \kappa^+$, such that the first class is D and any other class is obtained from classes preceding it by one of the operations. If $\kappa \in \mathrm{H}(D)$, by the lemma κ will be in every class of such a sequence.

By a system of operations (using F and starting from D) we mean (a class which codes) the following.

- A well order on Card, where this is a linear order with no descending chains of length ω.
- For each limit point of the well order whose cofinality is less than Ord, a cofinal map from an ordinal through the predecessors.
- For each limit point of the well order whose cofinality equals Ord, a cofinal map from Ord through the predecessors.
- For each point α of the well order a class S_α, such that $S_{\alpha'} = F(S_\alpha)$ (α' denoting the successor in the well order); and if α is a limit point in the well order, S_α is the intersection (if $\mathrm{Cf}(\alpha) <$ Ord) or diagonal intersection (if $\mathrm{Cf}(\alpha) =$ Ord) of the classes in the cofinal sequence.

Call the sequence of S_α the trace of the system.

It is a routine exercise to write down a Π^0_1 formula in the free class variables Ξ and D which states that Ξ is a system of operations starting from D, provided $Y = F(X)$ is Π^0_1 definable. This formula also defines a system of operations in V_κ for any $\kappa \in$ Regu.

LEMMA 6. If $D \subseteq \mathrm{Reg}$ and $\kappa \in D - \mathrm{H}(D)$ then there is a system of operations (using H) of length $< \kappa^+$ which removes all $\alpha \in D \cap \kappa$ (and hence one which removes κ).

PROOF: Clearly we may assume $\kappa > \omega$. Let $Z \subseteq \kappa$ be a club disjoint from D, and enumerate Z in natural order as $\langle \alpha_\gamma : \gamma < \kappa \rangle$. Let $Y_0 = D$ and for $\gamma < \kappa$ let $Y_{\gamma+1}$ be some subset of Y_γ obtainable by the operations, which is disjoint from $\alpha_{\gamma+1}$; clearly such exists. If γ is a limit ordinal let $Y_\gamma = \bigcap_{\delta<\gamma} Y_\delta$. The diagonal intersection of the Y_γ contains no elements below κ; for if $\gamma < \alpha_\gamma$ then clearly $\gamma \neq Y_\gamma$, and if $\gamma = \alpha_\gamma$ then $\gamma \notin D$.

THEOREM 7. Suppose $D \subseteq \text{Reg}$ and $\kappa \in D$; the following are equivalent.

a. $\kappa \in \text{H}(D)$.

b. $\kappa \neq \omega$ and if $X \subseteq \kappa$ is unbounded then $\text{Cl}(X) \cap D \neq \emptyset$.

c. $\kappa \neq \omega$ and if $X \subseteq \kappa$ is unbounded then $\text{Cl}(X) \cap D$ is a D-club.

d. $\kappa \in \text{Lim}(D)$ and the D-clubs are closed under Fix, intersections of length $< \kappa$, and continuous diagonal intersections of length κ.

e. $\kappa \in \text{Lim}(D)$ and the D-clubs are closed under Fix.

f. $\kappa \in \text{Lim}(D)$ and D-clubs are stationary.

PROOF: The equivalence of b and c to a is immediate. The equivalence of d to a follows by lemmas 5 and 6; note that $\kappa \in \text{Lim}(D)$ iff $D \cap \kappa$ is a D-club. To show that e implies a, suppose $\kappa \notin \text{H}(D)$, and let C be a club disjoint from D. Since D is unbounded, we may choose a continuous sequence λ_γ, $\gamma < \kappa$, such that for all $\gamma < \kappa$ $\lambda_{\gamma+1} \in D$ and there is an $\alpha \in C$ with $\lambda_\gamma < \alpha < \lambda_{\gamma+1}$. Deleting the λ_γ with $\gamma \in \text{Lim}$ yields a D-club with no limit points. To see that a implies f, if $\kappa \in \text{H}(D)$, and X is a D-club and Y a club then $D \cap \text{Cl}(X) \cap Y = X \cap Y$ is nonempty. To see that f implies a, if $\kappa \notin \text{H}(D)$ but $\kappa \in \text{Lim}(D)$ then $D \cap \kappa$ is a D-club, but is not stationary.

Call the filter in $D \cap \kappa$ generated by the D-clubs the D-club filter; this exists (i.e., is nonempty) if $\kappa \in \text{Lim}(D)$. The first part of the following theorem very likely falls in the category of folklore.

THEOREM 8. Suppose $D \subseteq \text{Reg}$ and $\kappa \in D$.

a. $\kappa \in \text{H}(D)$ iff there is a κ-complete normal filter in $D \cap \kappa$ closed under Fix.

b. If $\kappa \in \text{H}(D)$ then the least such filter F equals the D-club filter.

PROOF: If $\kappa \in \text{H}(D)$ then the D-club filter has the required properties by lemma 5. If $\kappa \in D - \text{H}(D)$ lemma 6 shows that there cannot be such a filter For part b, suppose $C = \langle \alpha_\gamma : \gamma < \kappa \rangle$ is a D-club. Let $X_{\gamma+1} = \{\delta \in D : \delta \geq \alpha_{\gamma+1}\}$, and for $\gamma \in \text{Lim}$ let $X_\gamma = \bigcap_{\delta<\gamma} X_\delta$. Clearly each X_γ is in F, hence so is $\triangle_\gamma X_\gamma$. Further for any $\gamma \in \triangle_\gamma X_\gamma$, $\gamma = \alpha_\gamma$ (use transfinite induction and the fact that C is a D-club), whence $\triangle_\gamma X_\gamma \subseteq C$, and $C \in F$.

The proof of theorem 8 shows that if $\kappa \in \text{H}(D)$ then the D-club filter is the least κ-complete normal filter in $D \cap \kappa$ containing the cobounded subsets of $D \cap \kappa$. Note also that if $F : \kappa \mapsto \kappa$ is monotone and $F[D \cap \kappa] \subseteq D \cap \kappa$ then there is a κ-complete normal filter closed under F in $D \cap \kappa$ iff there is one in κ containing $D \cap \kappa$.

THEOREM 9. Axiom W implies the axiom scheme

- if F is a Reg-club in Ord then $\text{Fix}(F)$ is;
- if F_γ, $\gamma < \beta$, is a Reg-club in Ord then $\bigcap_{\gamma<\beta} F_\gamma$ is;
- if F_γ, $\gamma \in \text{Ord}$, is a Reg-club in Ord then $\triangle_\gamma F_\gamma$ is, provided $\langle F_\gamma \rangle$ is continuous.

This in turn implies axiom F.

Proof: The first claim is proved by applying W to the hypothesis of each part, conjoined with $\alpha = \alpha$, to get arbitrarily large $\kappa \in$ Reg such that V_κ satisifes the hypothesis. Such a κ is in the class of the conclusion; it follows that the class of the conclusion is unbounded, and by the usual arguments, adapted to classes, it is closed in Reg. To show that axiom F follows, let F be a club in Ord, and suppose it is disjoint from Reg. Then we can adapt the argument of lemma 6 to classes, and define a system of operations yielding the empty class. On the other hand, using the intermediate axiom scheme we can show that every class in the trace of a definable system of operations must be a Reg-club in Ord. This contradiction shows that F is not disjoint from Inac.

It has been pointed out to the author that W⇒F is almost immediate. If C is a closed unbounded class, then the statement that it is holds in some V_κ, $\kappa \in$ Reg, and $\kappa \in C$ follows.

The existence of a Mahlo cardinal is (if consistent) independent of axiom R. If κ is the least Mahlo cardinal then there is a $\lambda < \kappa$ with $V_\lambda \prec V_\kappa$; indeed we may require $\lambda \in$ Reg. Adding the existence of a Mahlo cardinal does not imply the existence of more than one.

On the other hand, axiom R implies the existence of quite large inaccessibles. Indeed, any class reachable by a sequence of operations is a Reg-club in Ord. Given a first order characterization $c(\alpha; x)$ of a family of classes $C(x) = \{c(\alpha; x) : \alpha \in \text{Ord}\}$ of inaccessibles, the first order statement that $c(\alpha; x)$ exists can be proved by proving it for the second order definition, and showing that the definitions are equivalent. The argument can be given in ZFCT, by theorem 4. This method applies for example to the classes $I(\phi)$ of section 1. A first order definition may be given by relativizing the recursion for $I(\alpha; \phi) = \kappa$ to κ; this follows by refinements of arguments in [Ve08].

5. Standard models.

If M is a set (or an object of type 1 in ZFCT), it may be considered as a structure for the language of set theory, with $\in \restriction M$ as the membership relation. M is called an $\in$-structure; if it is a model of a theory T it is called a standard model of T. If the language is expanded with additional relations, these may be interpreted as subsets of M. As noted above, we use $M \prec N$ for "M is an elementary substructure of N"; we also use $M \equiv N$ for "M is elementarily equivalent to N".

Statements about standard models fall in to two categories, those which can be made in the language of set theory, and those which require a truth predicate, say for the first order formulas with parameters from the universe. An example in the first category is the statement that M is a standard model of ZFC. Refinements include requiring that M be a V_κ or L_α, or contain some set X or $X \in L$, or be transitive. Note that if M is a transitive model of ZFC and $M \cap \text{Ord} = \alpha$ then L_α is a model of ZFC. Examples in the second category are $M \prec V$ or $M \equiv V$, with the same refinements as above, and also $M \prec L$ or $M \equiv L$.

Statements in the first category follow from Tarski's axiom, and in some cases even from the existence of an inaccessible cardinal. Statements in the second category require second order machinery, which (usually) suffices to prove statements in the first category also.

The minimal second order machinery required consists of a truth predicate, its defining axioms TDF, a global choice function, and the axiom GC stating that it is one. In ZF+TDF+GC, a Skolem function for V can be defined and proved to be one; a Skolem function for L can also be shown to exist. From this, $\in$-structures M with $M \prec V$, etc., can be shown to exist. To prove that ZFC has a standard model it is only necessary to add the axiom TRP, which states the truth of the replacement axiom scheme; the truth of any sentance provable in ZFC also follows. An informal version of this argument may be found in [Co66]; see also [Dr74], exercise 5.3.4.2.

Indeed, it follows that the κ such that $V_\kappa \prec V$ form a club in Ord. If an appropriate version of axiom R is added to the system, it follows that the $\kappa \in$ Reg such that $V_\kappa \prec V$ form a Reg-club in Ord. It follows from Tarski's axiom that there is a club in Ord of κ such that V_κ is a model of ZFC; namely, those for which $V_\kappa \prec V_\lambda$, some $\lambda \in$ Inac. In this connection, in [MV59] it is shown that if $V_\alpha \prec V_\beta$ then both are models of ZFC.

Various observations can be made about V_κ or L_α which are standard models; see [MV59] and [Wi68] for some of these. If V_κ is a model of ZFC, κ must be a strong limit cardinal, and satisfy $|V_\kappa| = \kappa$. The latter implies $\aleph_\kappa = \kappa$; the converse implication follows using GCH. If κ is regular it is inaccessible. If $V_\kappa \prec V$ V_κ is closed under definability, and in particular contains the definable sets. The supertransitive models of ZFC coincide with the V_κ which are models.

If $L_\alpha \equiv V$ then $V{=}L$. If $L_\alpha \prec L$, it is closed under definability in L, and in particular contains the sets definable in L; in fact, $M \prec L$ iff M is closed under definability in L. If $|V_\kappa| = \kappa$ then $L_\kappa = V_\kappa^L$, so if $V_\kappa \prec V$ then $L_\kappa \prec L$. In particular, α can be a singular cardinal. If $V{=}L$ and L_κ is a model of ZFC then κ is a strong limit cardinal, so if $V{=}L$ then α can be a regular cardinal only if it is inaccessible. If $\kappa \in$ Regu then $\{\alpha < \kappa : L_\alpha \prec L_\kappa\}$ is a club. It follows that $L_\kappa \prec L$ iff $\{\alpha < \kappa : L_\alpha \prec L\}$ is a club; one direction is easy, and for the other, given $F(\vec{p})$, $\vec{p} \in L_\alpha$, $L_\alpha \prec L_\kappa, L$ it follows that $\models_L F(\vec{p}) \Leftrightarrow \models_{L_\alpha} F(\vec{p}) \Leftrightarrow \models_{L_\kappa} F(\vec{p})$.

Interest in standard models stems partly from their use in independence proofs using forcing. In these arguments, the existence of a standard model is a convenience; however if it exists then the generic model may also, and will if the generic filter exists. For the latter, we have the following theorem. For definitions of filter and so forth we follow [Je77]. Let $(P, <)$ be a partial order and κ an infinite cardinal. P is λ-closed if for every descending sequence of length at most λ there is an element $q \in P$ with $q \leq p$ for each element p of the sequence. P is $<\kappa$-closed if P is λ-closed for every infinite cardinal $\lambda < \kappa$.

Theorem 10. Suppose κ is regular, $(P, <)$ is a $<\kappa$-closed partial order, and D is a set of at most κ subsets of P. Then there is a filter $F \subseteq P$ which intersects every subset in D.

Proof: Enumerate D as D_α. Define a sequence p_α in P by transfinite induction; p_α is any member of D_α where $p_\alpha \leq p_\beta$ for $\beta < \alpha$. Let $F = \{q : q \geq p_\alpha,\ \text{some } \alpha\}$.

On the other hand if $\alpha > \omega_1$, L_α is a model of ZFC, and P is the finite sequences with values in $\omega_1^{L_\alpha}$, then it is independent whether a generic filter exists. As in [Je77], 1.46, it is consistent that one does, but if $\omega_1^L = \omega_1$ there is not one.

If the cardinality κ of α is regular, $\kappa < \alpha$, and L_α is a model of ZFC then generic extensions exist. For example, let P be the functions from initial segments of κ to $\{0,1\}$. If κ is uncountable and L_κ is a model Easton forcing provides a model which is a generic extension by a class. We leave the case of singular κ open.

Another topic of interest is standard models of consistent extensions of ZFC. These do not always exist, ZFC+"ZFC is not consistent" being an example. Even if they do, they may be restricted to have the same ordinals as the minimal model, ZFC+"ZFC has no standard model" being an example of this. A regularity condition on the theory T ruling out such pathologies would require the κ such that V_κ is a model of T to contain a club in Ord. Finally a theory T has a constructible standard model iff T+V=L has a standard model iff T has a model L_α, some α.

6. A stronger axiom.

To relativize a formula of ZFCT to an $\in$-structure D, the free variables must be relativized. Clearly, if X is a class then the relativized value X' of X should be $X \cap D$, since then for $x \in D$, $X'(x) = X(x)$. It is not so clear how higher order free variables should be relativized, so only class free variables are considered.

Let $T_{\alpha\beta}$ denote the formulas whose bound variables have type $< \alpha$ (with ∞ denoting no restriction), and whose free variables have type $\leq \beta$, $\beta = 0, 1$. For a class C of ordinals let $\mathrm{W}_{\alpha\beta}^C$ denote the axiom scheme

$$F \Rightarrow \exists\kappa \in C(\vec{x} \in V_\kappa \wedge F^{V_\kappa})$$

for the $T_{\alpha\beta}$ formulas F.

It is well known that V_κ satisfies W_{11} iff κ is inaccessible. It follows similarly that V_κ satisfies $\mathrm{W}_{11}^{\mathrm{Reg}}$ iff κ is Mahlo. W_{11} is provable in ZFC, suitably extended to allow free class variables [Gl73]. The cardinals such that V_κ satisfies W_{21} are those which are Π_n^1-indescribable for all n. We will call those κ such that V_κ satisfies $\mathrm{W}_{\alpha 1}$ T_α-indescribable, or T-indescribable if α=∞. It seems likely that T-indescribable cardinals exist. One would be more convinced if some method were known for "building up" the smallest Π_1^1-indescribable; we discuss this further below.

One might pursue what ordinals satisfy $\mathrm{W}_{\alpha 0}^C$ for various α, C; we content ourselves with a few observations. An ordinal α satisfies W_{10} iff V_α is a model of ZFC. In one direction, W_{10} is provable in ZFC. In the other, the axioms other than replacement are straightforward, noting that α must be a limit ordinal; replacement follows as in [Je77], lemma 32.2, since the required sentance for each definable function is first order. Finally, the least inaccessible does not satisfy W_{20}; the T_{20} sentance

$$\forall F \forall \alpha \text{"}F[\alpha] \text{ is bounded in Card and } |2^\kappa| \text{ exists for all } \kappa\text{"}$$

attests to this.

If κ is T_α-indescribable and F ranges over the $T_{\alpha 1}$ formulas such that $\models_{V_\kappa} F$, the sets

$$\{\gamma < \kappa : \vec{x} \in V_\gamma \wedge \models_{V_\kappa} F^{V_\gamma}\}$$

clearly generate a filter in κ, whose members we call T_α-enforceable. In the case of Π^1_1-indescribables, this filter is normal; in the case of T_α-indescribables, the usual proof of normality does not go through. However the sub-filter obtained by restricting F to the $T_{\beta 1}$ formulas for any $\beta < \alpha$ (or the Π^1_n formulas if $\alpha = 2$) is normal, as the usual proof (q.v. see [Dr74], theorem 9.2.4) shows, using the Tru predicate of section 3. Also, the usual proof ([Dr74], theorem 9.1.1) shows that the T_β-indescribables, $\beta < \alpha$, are T_α-enforceable.

Call a cardinal greatly Mahlo if there is a κ-complete normal filter in Inac $\cap\, \kappa$ which is closed under H. These or closely related cardinals are considered in [Ha64], [Ga67], [BTW77], [Gl73], and [Je84]. The smallest greatly Mahlo cardinal is an example of a cardinal which can be considered to have been "built up". Indeed, if the principle of collecting the universe is strengthened, to require that a universe have the Mahlo property, one can easily conclude that there are Mahlo κ below which the Mahlo cardinals are unbounded, indeed, form a Mahlo-club. It is worth noting that this does not mean they form an Inac-club; there is a $\lambda < \kappa$, $\lambda \in$ Inac, with $V_\lambda \prec V_\kappa$ where κ is the smallest Mahlo cardinal. Continuing as for inaccessibles builds up a greatly Mahlo cardinal.

THEOREM 11. The greatly Mahlo cardinals are Π^1_1-enforceable.

PROOF: It is shown in [Gl73] that the greatly Mahlo cardinals are Π^1_1-indescribable. The formula attesting to their enforceability is

$\forall W$ "If W is a system of operations then every class in its trace is nonempty"

where the system of operations uses H and starts at Inac.

This proof shows more; let $F^\nabla(X)$ be the operation where $\alpha \in F^\nabla(X)$ iff $\alpha \in Y$ for every class Y in the trace of any system of operations using F and starting at X.

THEOREM 12. The Π^1_1-enforceable classes are closed under H^∇.

PROOF: As in theorem 11, but starting from any Π^1_1-enforceable class X. If X is Π^1_1-enforceable then the formula is satisfied at V_κ for κ Π^1_1-indescribable; see [Dr74].

THEOREM 13. Let $\mathcal{F}$ be the collection of local functions under which the Π^1_1-enforceable filter is closed. Then $\mathcal{F}$ is closed under the operation $F \mapsto F^\nabla$.

PROOF: As usual, noting that a local function can be coded as a class.

This direction should be pursued, in an effort to gain an understanding of the size of the smallest Π^1_1-indescribable cardinal, but we will not do so here. It seems reasonable to claim that the smallest cardinal bearing a filter with with the property of theorem 13 has been built up.

[Gl73] considers the operation on classes $H_R(X) = \mathrm{Reg} \cap \mathrm{Hu}(X)$. If X is restricted to classes of regular cardinals this operation is identical to Hu. Indeed, if $\alpha \in \mathrm{Lim}$, $\mathrm{Cf}(\alpha) > \omega$, $\alpha \notin \mathrm{Reg}$, it is not difficult to see that there is a club of singular ordinals in α. Thus, $\mathrm{Hu}(\mathrm{Reg}) \subseteq \mathrm{Reg}$, whence if $X \subseteq \mathrm{Reg}$ then $\mathrm{Hu}(X) \subseteq \mathrm{Reg}$.

THEOREM 14. A cardinal κ is greatly Mahlo iff there is a κ-complete normal filter in $\mathrm{Inac} \cap \kappa$ which is closed under H_R.

PROOF: One direction follows since for $X \subseteq \mathrm{Reg}$, $\mathrm{H}(X) \subseteq \mathrm{H}_R(X)$. The converse follows because in a system of operations from Inac, the trace is the same whether H or H_R is used. Letting M_α denote the class using H, it suffices to show inductively that $\mathrm{Reg} \cap \mathrm{Hu}(M_\alpha) \subseteq M_\alpha$, which we leave to the reader (see the end of the section for some sublemmas).

Define κ to be G-Mahlo if $\{\mathrm{H}_R(X) : X \text{ stationary}\}$ generates a κ-complete normal filter closed under H_R. It follows as in [Gl73] that a Π^1_1-indescribable is G-Mahlo; the converse is independent and holds if $V{=}L$. Clearly, G-Mahlo cardinals are greatly Mahlo, but if $V{=}L$ the converse does not hold.

To conclude this section, we give a development of the rank function of [Je84]. In our context this rank is most meaningful for Mahlo cardinals. If one starts at Inac, the rank of κ is ≥ 1 iff κ is Mahlo, ≥ 2 iff it is hyper-Mahlo, etc. It is $\geq \kappa^+$ iff κ is greatly Mahlo. One can start at Reg, and get the weakly Mahlo cardinals, etc. If one uses Hu and starts at Ord, it is shown in [Je84] (and essentially in [BTW77]) that the rank of κ is $\geq \kappa$ iff κ is weakly inaccessible; and that the ordinals α with cofinality at least the βth regular cardinal comprise a maximal set of rank β.

If I is an ideal in the Boolean algebra of subsets of a set D, define $X \subseteq_I Y$ if $X - Y \in I$, and $X =_I Y$ if the symmetric difference $X \oplus Y$ is in I. The latter relation is a congruence relation whose quotient is the quotient Boolean algebra of the ideal. Suppose $\kappa \in \mathrm{Regu}$ and $D \subseteq \kappa$. If I is κ-complete, normal, and contains the bounded subsets of D, then $\bigcap_{\gamma<\alpha}$ for $\alpha < \kappa$ respects I; $\triangle_{\gamma<\kappa}$ respects I; and $\triangle_\gamma X_\gamma \subseteq_I X_\gamma$.

If $D \subseteq \mathrm{Ord}$ define a derivation operator on D to be a map from subclasses of D to subclasses of D, which is monotone, local, satisfies $F(F(X)) \subseteq F(X)$, and satisfies $F(\bigcup_{n<\omega} X_n) = \bigcup_{n<\omega} F(X_n)$. By locality, F determines a function on subsets of $D \cap \alpha$ for any $\alpha \in \mathrm{Ord}$, with the same properties. If (for $\kappa \in \mathrm{Regu}$ and I an ideal in $D \cap \kappa$) $F(X) \subseteq F(Y)$ when $X \subseteq_I Y$, say that F respects I. In such a case, define $X <_F Y$ for $X, Y \subseteq D \cap \kappa$ if $Y \subseteq_I F(X)$. Clearly, this relation is transitive; also if $W <_F Y$ and $X \subseteq_I Y$ then $W <_F X$.

Given any well-founded transitive relation $<$, a rank function ρ may be defined, by $\rho(X) = \sup\{\rho(W)+1 : W < X\}$. Clearly, if $W < X$ then $\rho(W) < \rho(X)$. It is readily shown by induction on $\rho(X)$ that if $\alpha < \rho(X)$ then there is a $W < X$ with $\rho(W) = \alpha$.

If $<_F$ is a well-founded transitive relation on the sets not in I, define M to be maximal of rank α if $\rho(M) = \alpha$ and for any X with $\rho(X) \geq \alpha$, $X \subseteq_I M$. Let M_α denote any such; clearly any two are equal mod I. The rank of κ is defined to be the sup of the ranks of the sets not in I. The following lemma is useful for showing that $<_F$ is well-founded.

LEMMA 15. If F is a derivation operator on D respecting I where I is countably complete, and if $\langle X_i : i \in \omega \rangle$ is a chain with $X_i \subseteq_I F(X_{i+1})$, then there is a chain $\langle Y_i \rangle$ with $Y_i =_I X_i$ and $Y_i \subseteq F(Y_{i+1})$.

PROOF: Define $X_i^0 = X_i$; $X_i^{j+1} = X_i^j$ for $i \geq j+1$; and $X_i^{j+1} = X_i^j - F(X_{i+1}^{j+1})$ for $i = j, \ldots, 0$ successively. It is readily verified that if A^j, B^j are such that $B^{j+1} \subset B^j$, $B^{j+1} - B^j \in I$, $A^0 \subseteq F(B^0)$, and $A^{j+1} = A^j - F(B^{j+1})$, then $\bigcap_j X^j \subseteq F(\bigcap_j B^j)$. The claim follows, letting $Y_i = \bigcap_j X_i^j$.

LEMMA 16. Suppose $<_F$ is a well-founded transitive relation, and M_α exists.

a. If $\alpha = \beta + 1$ then $M_\alpha = F(M_\beta)$.

b. If $\alpha = \sup\{\alpha_\delta\}$ for $\delta < \beta$ where $\beta < \kappa$ and $\beta \in \mathrm{Lim}$, and I is κ-complete, then $M_\alpha = \bigcap_{\delta<\beta} M_{\alpha_\delta}$.

c. If $\alpha = \sup\{\alpha_\delta\}$ for $\delta < \kappa$, and I is normal, then $M_\alpha = \triangle_\delta M_{\alpha_\delta}$.

PROOF: In all cases let M denote the right side. Clearly $\rho(M) \geq \alpha$; and if $\rho(M) > \alpha$ let $Y < M$ be such that $\rho(Y) = \alpha$. Then $M \subseteq_I F(Y)$, and a contradiction will be obtained if $Y \subseteq_I M$ can be shown. In case a, we can find $Z < Y$ with $\rho(Z) = \beta$; then $Y \subseteq_I F(Z)$, and $Z \subseteq M_\beta$, and $Y \subseteq_I M$ follows. In case b, $Y \subseteq_I M_{\alpha_\delta}$ because $\rho(Y) \geq \rho(M_{\alpha_\delta})$. Case c is similar.

It is routine to verify that H_R is a derivation operator on Reg; also, letting I be the complements in $\mathrm{Reg} \cap \kappa$ of the Reg-clubs, H_R respects I (see below). That $<_{\mathrm{H}_R}$ is well-founded follows easily by lemma 15. If κ is Mahlo, I is κ-complete and normal; also a maximal set of rank 0 exists (for this κ need only be 1-inaccessible).

For the proof of the claims for H_R, clearly it is monotone and local. Now, if $\mathrm{Cf}(\alpha) > \omega$ and $\mathrm{Hu}(X)$ is stationary then X is stationary. Indeed, let C be a club; then $\mathrm{Lim}(C)$ is a club, so there is some $\gamma \in \mathrm{Lim}(C) \cap \mathrm{Hu}(X)$. Since $\gamma \in \mathrm{Lim}(C)$, $C \cap \gamma$ is club in γ; and since $\gamma \in \mathrm{Hu}(X)$, $X \cap \gamma$ is stationary in γ. Thus, $C \cap X$ is nonempty, and since C was arbitrary X is stationary. It follows readily that $\mathrm{H}_R(\mathrm{H}_R(X)) \subseteq \mathrm{H}_R(X)$; if $\kappa \in \mathrm{Reg} \cap \mathrm{Hu}(\mathrm{H}_R(X))$ then $\mathrm{H}_R(X)$ is stationary in κ, so $\mathrm{Hu}(X)$ is, so X is, so $\kappa \in \mathrm{Reg} \cap \mathrm{Hu}(X) = \mathrm{H}_R(X)$. Next, $\bigcup_i \mathrm{H}_R(X_i) \subseteq \mathrm{H}_R(\bigcup_i X_i)$ by monotonicity. If $\alpha \in \mathrm{Hu}(\bigcup_i X_i)$ then $\bigcup_i X_i$ is stationary in α, so some X_i is, so $\alpha \in \mathrm{Hu}(X_i)$; that is, the opposite inclusion holds also. If $X - Y$ is thin so is $\mathrm{H}_R(X - Y)$, so it suffices to show that $\mathrm{H}_R(X) - \mathrm{H}_R(Y) \subseteq \mathrm{H}_R(X - Y)$; this follows from $\mathrm{H}_R(X - Y) = \mathrm{H}_R(X) - \mathrm{H}_R(X \cap Y)$.

7. Remarks on constructibility.

The hypothesis of constructibility, that all sets are constructible, has proved to be a very powerful principle. In the universe L of constructible sets many powerful combinatorial facts can be proved by logical arguments, and many statements proved which can be shown to be independent of ZFC. Some set theorists have rejected the hypothesis of constructibility on the grounds that there is no evidence for it in the cumulative hierarchy picture of sets, and see the constructible sets as an inner model about which we know more than we know about the real universe V, because L is simpler than V.

Upon further consideration, it is easy to see that this position is not critical enough. The possibility of the truth of $V{=}L$ has not been adequately considered, and indeed there are considerations which indicate that its truth is not as unlikely as has heretofore been supposed. The most obvious of these is that the axioms of ZFC contain a sufficient amount of information about sets that V should not contain any transitive standard inner model which contains Ord. Also, sets are quite simple, and it is entirely plausible that there is a set of simple operations which constructs the power set of a set if iterated through a sufficiently large ordinal.

The axiom of choice is the most basic consequence of constructibility. Most mathematicians would probably grant the truth of choice, indeed of strong choice. Its paradoxical consequences are facts of mathematical reality, and it should not be surprising that the latter should be full of unexpected subtleties. The existence of a set which is not Lebesgue measurable, for example, should be seen as a mathematical limitation on the scope of the definition of the measure of a set. The existence of such a set follows from the mere fact that a subgroup of a group has a system of coset representatives, and such is a plainly true fact about sets.

The principal reason that the axiom of choice (AC) is set apart from the remaining axioms is its "nonconstructive" aspect. The defining property of a choice function does not specify it uniquely. The independence of AC from ZF suggests that this is a fact about sets inherent in their nature; the arbitrary choice cannot be done away with in general. On the other hand, it is quite clear that such a choice can be made. AC may be considered to be a nonconstructive existence principle, and its independence shows that such principles are necessary in forging an understanding of the world of sets.

The continuum hypothesis (CH) is a slighty less immediate consequence of $V{=}L$. CH can be considered a nonconstructive existence principle, namely the existence of a well-ordering of the power set of $\aleph_0$ by $\aleph_1$. CH cannot be claimed to be clearly true, as AC can; however, it can be argued (and we believe) that it is true. Firstly, there is no "obvious" mathematical object whose cardinality is between $\aleph_0$ and c; indeed it is independent of ZFC whether there is one. One would surely expect that if such objects existed they would be apparent. Secondly, it seems reasonable that there are more types of well-orderings of $\aleph_1$ than there are subsets of $\aleph_0$. Indeed, there is no embedding via well-orderings, as is the case for well-orderings of $\aleph_0$.

There is little evidence that CH is false. It has been claimed that the inability to demonstrate a well-ordering is such evidence, but this can be seen as showing only that the well-ordering is nonconstructive, in that its existence is independent of ZFC. That a well ordering does exist is not surprising; indeed the independence of its existence is not. There is no hint of a contrary existence principle of this character. Such a principle would have to specify which $\aleph_\alpha$ equalled c, and there seems little doubt that $\alpha = 1$. The nonconstructive existence principle settling CH in the affirmative is entirely plausable, whereas there is no candidate for such settling it in the negative.

From $V{=}L$ it follows that there are quite canonical well orderings of the continuum by $\aleph_1$, wherein the reals are ranked according to how often the constructing operations must be applied to obtain them. This has been seen as evidence against constructibility, but again if the sets are as simple as constructibility indicates, then such an ordering is possible, and indeed is provided by the construction of the sets.

Similar reasoning leads one to conclude that GCH is probably true. The next most complex existence principle which follows from $V{=}L$ is diamond. There is less evidence for this than for CH; however there is one relevant result. If one assumes CH then diamond follows from a slightly weaker principle; see exercise III.3D of [De84].

$V{=}L$ is a "master" nonconstructive existence principle, asserting the existence of a construction sequence for each set. It is clear that its possible truth should be given careful consideration. The crux of the matter is that every subset of a set must eventually be constructed. This is a very strong version of the statement that the subsets can be ordered by a specific ordinal; indeed, they will all appear if one generates subsets in a certain manner. The cumulative hierarchy provides no justification of this, but this is not surprising; the cumulative hierarchy provides no justification for CH. What is required is exactly that some nonconstructive existence principles be added. All the ones which there is any reason to suspect might be true, namely AC, GCH, and diamond, are among the more basic consequences of $V = L$.

Whether $V{=}L$ is true is obviously a deep matter; however evidence that this is so is not as lacking as has sometimes been supposed, and we believe that it is true. Obviously a more complete understanding of which large cardinals can be "built up" would be valuable, especially if these turn out to be more or less the same as those that do not contradict the hypothesis of constructibility. From the point of view suggested here, one suspects that the existence of measurable cardinals is consistent, but false. It is obviously of great interest whether $\kappa(\omega)$ can be built up.

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