SCHEMES, ORDINAL FUNCTIONS, AND REPEAT POINTS

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Abstract: Schemes may be used to define systems of ordinal functions. Such systems may be used to obtain lower bounds on the smallest repeat point. Using a scheme Σ_{IVT}, it is shown that the smallest ordinal θ_T such that there is no T-separating set at θ_T is at least an ordinal which is larger than the large Veblen ordinal for the infinitary Veblen function starting at the critical point enumerator C. This holds for any coherent sequence satisfying $o(\mathcal{U}(\kappa)(\beta)) = \beta$. The notion of an O-scheme in $L[\mathcal{U}]$ is defined, and used to give sufficient conditions for non-existence of repeat points. A function W is defined in $L[\mathcal{U}]$, and it is shown using C and W that in $L[\mathcal{U}]$, if $\theta < \theta_T$ then $\mathrm{Pow}(\kappa) \not\subseteq L_\theta[\mathcal{U}]$.

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1. Introduction

Schemes were defined by the author in [4], where they were called systems of operations. They have been used subsequently by the author for various purposes, in [5], [6], [7], [8], [9], [10], [11]. Here they will be used to define systems of ordinal functions.

Ordinal function is a term used in ordinal notation theory, for functions which are used to define ordinals from smaller ones. An early example is the binary and infinitary Veblen functions of [25]. More recent examples may be found in [1], [22], [24]; [2] has a survey of some systems.

Here, schemes are used to obtain systems of ordinal functions. The question of obtaining lower bounds on the smallest repeat point at a measurable cardinal using such systems is considered.

For later use, recall from [21] that an ordinal α is an ϵ-number iff $\alpha = 1$ or α is closed under ordinal exponentiation.

2. Schemes

Suppose Ω is a regular uncountable cardinal. For a sequence $\langle X_\xi : \xi < \Omega\rangle$ of subsets $X_\xi \subseteq \Omega$, define the following three variations of the diagonal intersection operation:

- $\triangle^{<}_{\xi<\Omega} X_\xi = \{\alpha : \forall \xi < \alpha(\alpha \in X_\xi)\}$.
- $\triangle^{\leq}_{\xi<\Omega} X_\xi = \{\alpha : \forall \xi \leq \alpha(\alpha \in X_\xi)\}$.
- $\triangle^{=}_{\xi<\Omega} X_\xi = \{\alpha : \alpha \in X_\alpha\}$.

It is readily seen that if each X_ξ is a club subset then $\triangle^{<}_{\xi<\Omega} X_\xi$ is club.

Say that $\langle X_\xi : \xi < \Omega\rangle$ is continuous if for $\xi \in \mathrm{Lim}$, $\cap_{\xi'<\xi} X_{\xi'} = X_\xi$.

Lemma 1. *If $\langle X_\xi : \xi < \Omega\rangle$ is continuous then $\triangle^{<}_{\xi<\Omega} X_\xi = \triangle^{\leq}_{\xi<\Omega} X_\xi \cup \{0\}$.*

Proof. This follows readily from the definitions. □

In particular, if $\langle X_\xi : \xi < \Omega\rangle$ is continuous, then if each X_ξ is a club subset then $\triangle^{\leq}_{\xi<\Omega} X_\xi$ is club. The operation $\triangle^{<}$ has the advantage that continuity is not required for the diagonal intersection of club subsets to be club. It has the disadvantage that the diagonal intersection always contains 0. As will be seen below, $\triangle^{\leq}$ can be advantageous.

Say that $\langle X_\xi : \xi < \Omega\rangle$ is descending if $\xi' \leq \xi$ implies $X_\xi \subseteq X_{\xi'}$.

Lemma 2. *If $\langle X_\xi : \xi < \Omega\rangle$ is descending then $\triangle^{=}_{\xi<\Omega} X_\xi = \triangle^{\leq}_{\xi<\Omega} X_\xi$.*

Proof. This follows readily from the definitions. □

Varying the definition of [11] slightly, a scheme, of length σ, over Ω, is defined to be a pair $\Sigma = \langle \sigma, \phi \rangle$ where $\sigma < \Omega^+$ and ϕ is a function whose domain is the set of limit ordinals $\alpha < \sigma$. For $\alpha \in \mathrm{Dom}(\phi)$, $\phi(\alpha)$ is an increasing function with domain an ordinal η, and whose range is an unbounded subset of α. If $\mathrm{Cf}(\alpha) < \Omega$ then $\eta < \Omega$ must hold, and if $\mathrm{Cf}(\alpha) = \Omega$ then $\eta = \Omega$ must hold.

A scheme Σ may be used to iterate an operation $F : \mathrm{Pow}(\Omega) \mapsto \mathrm{Pow}(\Omega)$ on a subset $X \subseteq \Omega$. For $o =<, \leq, =$, for a scheme Σ, for $\alpha < \sigma$, define $F^{\Sigma^o,\alpha}(X)$ recursively as follows.

0. $F^{\Sigma^o,0}(X) = X$.
1. If $\alpha = \beta + 1$ then $F^{\Sigma^o,\alpha}(X) = F(F^{\Sigma^o,\beta}(X))$.
2. If $\alpha \in \mathrm{Lim}$, $\mathrm{Cf}(\alpha) < \Omega$, and $\mathrm{Dom}(\phi) = \eta$ then $F^{\Sigma^o,\alpha}(X) = \cap_{\xi<\eta} F^{\Sigma^o,\phi(\alpha)(\xi)}(X)$.
3. If $\alpha \in \mathrm{Lim}$ and $\mathrm{Cf}(\alpha) = \Omega$ then $F^{\Sigma^o,\alpha}(X) = \triangle^o_{\xi<\Omega} F^{\Sigma^o,\phi(\alpha)(\xi)}$ (X).

It is readily seen that if X is a club subset and F maps club subsets to club subsets then $F^{\Sigma^<,\alpha}(X)$ is club for all α. This is also true for $F^{\Sigma^\leq,\alpha}(X)$, provided ascending sequences at α of cofinality Ω are continuous.

Lemma 3. *Suppose $F(X) \subseteq X$ for all X. For $\alpha \geq 1$, $F^{\Sigma^<,\alpha}(X) \subseteq F(X) \cup \{0\} \cup (\{1\} \cap X)$.*

Proof. By induction on α; details are left to the reader. $\square$

3. Example 1: Schemes for IV functions

Veblen [25] defined the infinitary Veblen (IV) hierarchy for an arbitrary uncountable cardinal, although regularity should be assumed. (According to [16] regularity was clarified by Hausdorff in 1908). A self-contained treatment was given in [11]. Some additional facts will be given here, relevant to the purposes of this paper.

Again let Ω be a regular uncountable cardinal (often λ^+ for some cardinal λ, such as ω).

Let $\mathcal{A}$ be the set of ordinal valued sequences $\alpha_0, \ldots, \alpha_\ell$ for some $\ell < \Omega$, such that $\alpha_\xi < \Omega$ for each $\xi \leq \ell$, only finitely many α_ξ are nonzero, and $\alpha_\ell > 0$ if $\ell > 0$. $\mathcal{A}$ may be ordered in reverse lexicographic order, where if $\bar{\alpha} = \alpha_0, \ldots, \alpha_\ell$ and $\bar{\alpha}' = \alpha'_0, \ldots, \alpha'_{\ell'}$, then $\bar\alpha' <_{\mathrm{rl}} \bar\alpha$ iff $\ell' < \ell$, or $\ell' = \ell$, and for some $\gamma \le \ell$, $\alpha'_\gamma < \alpha_\gamma$ and $\alpha'_\xi = \alpha_\xi$ for $\gamma < \xi \le \ell$. This relation is well-known to be a well-order (see [25] for example). The order type of $<_{\mathrm{rl}}$ is well-known to be Ω^Ω (see section 16 of [23]).

A scheme Σ_{IV} of length Ω^Ω may be defined by defining $\phi_{\bar\alpha}$ for each $\bar\alpha$ which is a limit point of $<_{\mathrm{rl}}$. Let ν denote the least index of a nonzero element α_ν. The definition breaks into the following cases, where η denotes $\mathrm{Dom}(\phi(\bar\alpha))$ and $\bar\alpha_\xi$ denotes $\phi(\bar\alpha)(\xi)$.

1. $\alpha_\nu = \beta+1$, $\nu = \gamma+1$: $\eta = \Omega$, $\bar\alpha_\xi = S_1(\bar\alpha, \xi)$ where in $S_1(\bar\alpha, \xi)$, α_ν is replaced by β and α_γ by ξ.
2. $\alpha_\nu = \beta+1$, $\nu \in \mathrm{Lim}$: $\eta = \nu$, $\bar\alpha_\xi = S_2(\bar\alpha, \xi)$ where in $S_2(\bar\alpha, \xi)$, α_ν is replaced by β and α_ξ by 1.
3. $\alpha_\nu \in \mathrm{Lim}$: $\eta = \alpha_\nu$, $\bar\alpha_\xi = S_3(\bar\alpha, \xi)$ where in $S_3(\bar\alpha, \xi)$, α_ν is replaced by ξ.

For a club R (R_α, etc.) let $\psi = \mathrm{Enum}(R)$ (ψ_α, etc.) be its normal enumerating function, so that $R = \mathrm{Ran}(\psi)$. Let $\mathrm{Fix}(\psi)$ denote the fixed point enumerator of ψ and let $\mathrm{Fix}(R)$ denote $\mathrm{Ran}(\mathrm{Fix}(\mathrm{Enum}(R)))$. The reader is assumed to be familiar with basic properties of Fix, e.g., that it takes club subsets to club subsets.

The infinitary Veblen functions $\psi_{\bar\alpha}$ may be specified using Σ_{IV}. Let R_0 be any club subset of Ω. Identifying $\bar\alpha$ with its ordinal position in $<_{\mathrm{rl}}$, $R_{\bar\alpha}$ equals $\mathrm{Fix}^{\Sigma^{\le}_{\mathrm{IV}}, \bar\alpha}(R_0)$.

Theorem 4. *Write an element $\bar\alpha \in \mathcal{A}$ as $\bar 0 \alpha_\nu \bar\delta$.*
a. The sequence $R_{\bar 0 \xi \bar\delta}$ is continuous.
b. $R_{\bar\alpha}$ is a club subset of Ω.

Proof. Part a follows by case 3 of the definition of Σ_{IV}. Part b follows by part a and lemma 1, by case 1 of the definition. □

Some additional facts are given in the next theorem. These can be used to verify that the definition given here yields the same $\psi_{\bar\alpha}$ as the more traditional definition given in [11]. (Note that lemma 4 of [11] is incorrect.)

Theorem 5. *Let notation be as in the preceding theorem.*
a. The sequence $R_{\bar 0 \xi \bar\delta}$ is descending.
b. In case 1 of the definition, $R_{\bar\alpha} = \triangle^{=}_{\xi} R_{S_1(\bar\alpha, \xi)}$.
c. If $\xi' < \xi$ then $R_{\bar 0 \xi \bar\delta} \subseteq Fix(R_{\bar 0 \xi' \bar\delta})$.

d. If $\xi' < \xi$ *then for any* ζ $\psi_{\bar{0}\xi'\beta}(\psi_{\bar{0}\xi\bar{\delta}}(\zeta)) = \psi_{\bar{0}\xi\bar{\delta}}(\zeta)$*.*

e. The sequence $\psi_{\bar{0}\xi\bar{\beta}}(0)$ *is normal.*

f. In case 1 of the definition, $R_{\bar{\alpha}}$ *equals Fix*$\{\psi_{S_1(\bar{\alpha},\xi)}(0) : \xi < \Omega\})$*.*

Proof. For part a, in case 1 of the definition, $R_{\bar{0}0,\beta+1,\bar{\delta}} = \triangle^{\leq}_{\xi} R_{\bar{0}\xi\beta\bar{\delta}} \subseteq R_{\bar{0}0\beta\bar{\delta}}$; and in case 2 $R_{\bar{0},\beta+1,\bar{\delta}} = \cap_{\gamma<\nu} R_{\bar{0}1_\gamma\bar{0}\beta\bar{\delta}} \subseteq R_{\bar{0}0\beta\bar{\delta}}$, where the inclusion follows inductively.

Part b follows by part a and lemma 2, by case 1 of the definition.

For part c, first note that it follows by induction that $0 \notin R_{\bar{\alpha}}$ for any $\bar{\alpha}$. It suffices to show $R_{\bar{0},\beta+1,\bar{\delta}} \subseteq \mathrm{Fix}(R_{\bar{0}\beta\bar{\delta}})$. This will be shown by induction on ν. The claim is readily verified when $\nu = 0$. At a successor ordinal case 1 applies. Inductively $R_{\bar{0}1\beta\bar{\delta}} \subseteq \mathrm{Fix}(R_{\bar{0}0\beta\bar{\delta}})$, and since $0 \notin \triangle^{\leq}_{\xi} R_{\bar{0}\xi\beta\bar{\delta}}$, $\triangle^{\leq}_{\xi} R_{\bar{0}\xi\beta\bar{\delta}} \subseteq R_{\bar{0}1\beta\bar{\delta}}$. At a limit ordinal case 2 applies. Inductively, $R_{\bar{0}1_\gamma\bar{0}\beta\bar{\delta}} \subseteq \mathrm{Fix}(R_{\bar{0}\beta\bar{\delta}})$, whence $R_{\bar{0},\beta+1,\bar{\delta}} \subseteq \cap_{\gamma<\nu} R_{\bar{0}1_\gamma\bar{0}\beta\bar{\delta}} \subseteq \mathrm{Fix}(R_{\bar{0}\beta\bar{\delta}})$.

Part d follows by part c.

For part e, it follow readily by part c that $\psi_{\bar{0}\xi\bar{\beta}}(0)$ is increasing. Suppose $\xi \in \mathrm{Lim}$; let $\chi = \sup_{\xi'<\xi} \psi_{\bar{0}\xi'\bar{\beta}}(0)$. For $\xi' < \xi$, $\psi_{\bar{0}\xi'\bar{\beta}}(\chi) = \psi_{\bar{0}\xi'\bar{\beta}}(\sup_{\xi''<\xi} \psi_{\bar{0}\xi''\bar{\beta}}(0)) = \sup_{\xi''<\xi} \psi_{\bar{0}\xi'\bar{\beta}}(\psi_{\bar{0}\xi''\bar{\beta}}(0)) = \sup_{\xi'<\xi''<\xi} \psi_{\bar{0}\xi'\bar{\beta}} (\psi_{\bar{0}\xi''\bar{\beta}}(0)) = \sup_{\xi'<\xi''<\xi} \psi_{\bar{0}\xi''\bar{\beta}}(0) = \chi$. Thus, $\chi \in R_{\bar{0}\xi\bar{\beta}}$, and since $\chi \leq \psi_{\bar{0}\xi\bar{\beta}}(0)$, $\chi = \psi_{\bar{0}\xi\bar{\beta}}(0)$. This shows $\psi_{\bar{0}\xi\bar{\beta}}(0)$ is continuous.

For part f, $\zeta \in \triangle^{=}_{\xi} R_{S_1(\bar{\alpha},\xi)}$ iff $\zeta \in R_{S_1(\bar{\alpha},\zeta)}$ iff $\zeta = \psi_{S_1(\bar{\alpha},\zeta)}(\xi)$ for some ξ. Since $\psi_{S_1(\bar{\alpha},\zeta)}(0)$ is increases with ζ, $\psi_{S_1(\bar{\alpha},\zeta)}(0) \geq \zeta$ and hence $\psi_{S_1(\bar{\alpha},\zeta)}(\xi) > 0$ if $\xi > 0$. Thus, $\zeta = \psi_{S_1(\bar{\alpha},\zeta)}(\xi)$ for some ξ iff $\zeta = \psi_{S_1(\bar{\alpha},\zeta)}(0)$, iff $\zeta \in \mathrm{Fix}(\{\psi_{S_1(\bar{\alpha},\xi)}(0)\})$. □

It requires more work to prove that the $R_{\bar{\alpha}}$ are club subsets when the operation $\triangle^{\leq}$ is used than when $\triangle^{<}$ is. The former has advantages, though.

The closure ordinal $\theta_{\mathrm{IV},\psi_0}$ of Σ_{IV} with initial function ψ_0 is defined to be the smallest ordinal θ such that if $\ell < \theta$, $\alpha_\xi < \theta$ for $\xi < \ell$, and $\zeta < \theta$, then $\psi_{\bar{\alpha}}(\zeta) < \theta$; and containing 0, although for some ψ_0 the initial subset might be larger.

When $\Omega = \mu^+$ where μ is a cardinal, and $\psi_0(\zeta) = \mu^\zeta$, the closure ordinal (which is called the large Veblen ordinal when $\mu = \omega$) has an alternative characterization, which will be useful in the next section.

As in [11], a CNF function is a function $C_k(\eta_k, \sigma_k, \ldots, \eta_1, \sigma_1)$, where $k \geq 1$. If $\eta_k > \cdots > \eta_1$ and $0 < \sigma_i < \Omega$ the arguments are said to be proper, and the value of C_k is $\mu^{\eta_k} \cdot \sigma_k + \cdots + \mu^{\eta_1} \cdot \sigma_1$. If the arguments are improper, the value is defined to be 0.

An infinitary Veblen (IV) function is a function $\psi_k(\zeta, \xi_1, \alpha_1, \ldots, \xi_k, \alpha_k)$ where $k \geq 1$. If $\xi_1 < \cdots < \xi_k$ and $\alpha_i > 0$ the arguments are said to be proper; the value of ψ_k is $\psi_{\bar{\alpha}}(\zeta)$ where $\bar{\alpha}$ is the sequence where $\alpha_{\xi_i} = \alpha_i$ (α_ξ is 0 elsewhere). If the arguments are improper the value is 0.

Lemma 6. *The closure ordinal θ_{IV,μ^ζ} is the smallest ordinal which contains* 0 *and is closed under CNF and IV functions.*

Proof. It suffices to show that if $\theta > 0$ is closed under μ^ζ where $\mu \geq 2$ then θ is closed under $+, \cdot$. Clearly $1, 2 < \theta$. Since μ^ζ is normal, $\gamma \cdot 2 \leq (\mu^\gamma)^2$, so if $\gamma < \theta$ then $\gamma \cdot 2 < \theta$. It follows that θ is closed under $+$. If $\alpha, \beta < \theta$ then $\alpha \cdot \beta \leq \mu^{\alpha+\beta} < \theta$. $\square$

4. IV terms

Suppose $\Omega = \mu^+$ for a cardinal μ. Consider the system of terms $\boldsymbol{\tau}$, whose leaves are ordinals $\sigma < \mu$, and whose interior nodes are either a CNF function C_k in the base μ, or an infinitary Veblen function ψ_k starting at μ^ζ. Call these the IV terms, in the base μ. A proper term is one where the arguments are proper at each interior node. A normal form term is a proper term where at each interior note, the value of the function is greater than the value of any of its arguments.

Clearly the value τ of a normal IV term $\boldsymbol{\tau}$ satisfies $\tau < \theta_{\mathrm{IV},\mu^\zeta}$. It is shown in theorem 9 of [11] that for each $\tau < \theta_{\mathrm{IV},\mu^\zeta}$ there is a unique normal form term $\boldsymbol{\tau}$ whose equals τ. This fact is well-known when $\mu = \omega$.

Indeed, it is well-known that a non-zero ordinal has a unique Cantor normal form in the base μ. Corollary 7 of [11] states that if $\tau < \theta_{\mathrm{IV},\mu^\zeta}$ is a fixed point of μ^ζ then there are a unique k and values $\zeta, \xi_1, \alpha_1, \ldots, \xi_k, \alpha_k < \tau$ such that $\tau = \psi_k(\zeta, \xi_1, \alpha_1, \ldots, \xi_k, \alpha_k)$.

Using these facts the above cited theorem follows by induction on τ. The induction is broken into four cases, according to the Cantor normal form $\mu^{\eta_k} \cdot \sigma_k + \cdots + \mu^{\eta_1} \cdot \sigma_1$ for τ.

0. $\tau = 0$.
1. $k = 1$ and $\eta_1 = 0$.

2. $k > 1$ or $\sigma_1 > 1$ or $\eta_1 > 0$ and $\eta_1 < \Omega^{\eta_1}$.
3. $k = 1$ and $\sigma_1 = 1$ and $\eta_1 = \Omega^{\eta_1}$.

Case 0 is separate from case 1 because CNF has only been defined for $\tau > 0$. These cases will be referred to below.

Let τ denote the value of the IV term $\boldsymbol{\tau}$. Let $\mathrm{O}(\tau_1, \tau_2)$ be the function where $\mathrm{O}(\tau_1, \tau_2)$ equals -1 if $\tau_1 < \tau_2$, 0 if $\tau_1 = \tau_2$, and $+1$ if $\tau_1 > \tau_2$. Let $\mathrm{P}(\boldsymbol{\tau})$ equal 1 if $\boldsymbol{\tau}$ is proper, else 0. Let $\mathrm{N}(\boldsymbol{\tau})$ equal 1 if $\boldsymbol{\tau}$ is normal, else 0.

Lemma 10 of [11] states that $\mathrm{O}(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ depends only on the values $\mathrm{O}(\sigma_1, \sigma_2)$ and $\mathrm{P}(\sigma)$ for the leaves of $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$; $\mathrm{P}(\boldsymbol{\tau})$ depends only on the values $\mathrm{O}(\sigma_1, \sigma_2)$ and $\mathrm{P}(\sigma)$ for the leaves of $\boldsymbol{\tau}$; and $\mathrm{N}(\boldsymbol{\tau})$ depends only on the values $\mathrm{O}(\sigma_1, \sigma_2)$ and $\mathrm{P}(\sigma)$ for the leaves of $\boldsymbol{\tau}$.

This gives rise to an algorithm (a "comparison oracle Turing machine" as defined in [12]) for computing O. This algorithm is uniform in μ and runs in polynomial time.

An ascending sequence for each limit ordinal $\tau < \theta_{\mathrm{IV},\mu^\zeta}$ may be specified by recursion on its term $\boldsymbol{\tau}$. This method goes back to [25]; the treatment of [11] is repeated here for convenience, and proofs given. The convention will be used of identifying a term $\boldsymbol{\tau}$ with its value τ. The notation τ_ξ will be used for $\phi(\tau)(\xi)$, and D_τ, or just D, for $\mathrm{Dom}(\phi(\tau))$. Recall the division into cases of τ given above.

Case 0 is irrelevant.

Case 1. Let $D = \sigma$ and $\tau_\xi = \xi$.

Case 2 is divided into subcases according to the CNF of τ.

Case 2.1, $k > 1$. Write τ as $\tau_1 + \tau_2$ where $\tau_2 = \mu^{\eta_1} \cdot \sigma_1$. $D = D_{\tau_2}$ and $\tau_\xi = \tau_1 + \tau_{2\xi}$.

Case 2.2, $\tau = \mu^\eta \cdot \sigma$ where $\sigma \in \mathrm{Lim}$. $D = D_\sigma$ and $\tau_\xi = \mu^\eta \cdot \sigma_\xi$.

Case 2.3, $\tau = \mu^\eta$ where $\eta \in \mathrm{Lim}$ and $\eta < \mu^\eta$. $D = D_\eta$ and $\tau_\xi = \mu^{\eta_\xi}$.

Case 2.4, $\tau = \mu^\eta$ where $\eta = \eta^- + 1$. $D = \mu$ and $\tau_\xi = \mu^{\eta^-} \cdot \xi$.

Case 3 is divided into subcases according to the IV function $\psi_k(\zeta, \pi_1, \alpha_1, \ldots)$. Cases will be denoted XYZ where X is the type of ζ (L for limit, 0, or S for successor), Y is the type of α_1 (L or S), and Z is the type of π_1 (L, 0, or S); in some cases Y or Z may be *, denoting any possibility.

Case 3.L**: $D = D_\zeta$ and $\tau_\xi = \psi_k(\zeta_\xi, \pi_1, \alpha_1, \ldots)$.

Case 3.0L*: $D = D_\alpha$ and $\tau_\xi = \psi_k(0, \pi_1, \alpha_{1\xi}, \ldots)$.

Case 3.SL*: $D = D_{\alpha_1}$ and $\tau_\xi = \psi_k(\gamma, \pi_1, \alpha_{1\xi}, \ldots)$ where $\gamma = \psi_k(\zeta^-, \pi_1, \alpha_1, \ldots) + 1$.

Case 3.0SL: $D = D_{\pi_1}$ and $\tau_\xi = \psi_{k+1}(0, \pi_{1\xi}, 1, \pi_1, \alpha_1^-, \ldots)$.

Case 3.SSL: $D = D_{\pi_1}$ and $\tau_\xi = \psi_k(\gamma, \pi_{1\xi}, \alpha_1^-, \ldots)$ where $\gamma = \psi_k(\zeta^-, \pi_1, \alpha_1, \ldots) + 1$.

Case 3.0S0: $D = \omega$, $\tau_0 = \psi_k(0, 0, \alpha_1^-, \ldots)$, $\tau_{n+1} = \psi_k(\tau_n, 0, \alpha_1^-, \ldots)$.

Case 3.SS0: $D = \omega$, $\tau_0 = \psi_k(\zeta^-, 0, \alpha_1, \ldots) + 1$, $\tau_{n+1} = \psi_k(\tau_n, 0, \alpha_1^-, \ldots)$.

Case 3.0SS: $D = \omega$, $\tau_0 = \psi_k(0, \pi_1, \alpha_1^-, \ldots)$, $\tau_{n+1} = \psi_{k+1}(0, \pi_1^-, \tau_n, \pi_1, \alpha_1^-, \ldots)$.

Case 3.SSS: $D = \omega$, $\tau_0 = \psi_k(\zeta^-, \pi_1, \alpha_1, \ldots)$, $\tau_{n+1} = \psi_{k+1}(0, \pi_1^-, \tau_n, \pi_1, \alpha_1^-, \ldots)$.

Lemma 7. *τ_ξ is an increasing sequence whose limit is τ.*

Proof. The following preliminary facts are readily verified. If $\psi : \Omega \mapsto \Omega$ is a normal function and $\langle \alpha_\xi : \xi < \eta \rangle$ where $\eta < \Omega$ ascends to $\alpha < \Omega$ then $\psi(\alpha_\xi)$ ascends to $\psi(\alpha)$. If $\langle R_\xi : \xi < \Omega \rangle$ is a descending chain of club subsets of Ω and $R_{\xi+1} \subseteq \mathrm{Fix}(R_\xi)$ then ψ_ξ is increasing. If in addition R_ξ is continuous then ψ_ξ is normal.

Case 1 is immediate. All subcases of case 2 follow by induction and ordinal arithmetic. Case 3.L** follows because $\zeta \mapsto \psi_k(\zeta, \pi_1, \alpha_1, \ldots)$ is normal. Case 3.0L* follows because $\alpha_1 \mapsto \psi_k(0, \pi_1, \alpha_1, \ldots)$ is normal.

For case 3.SL*, writing ψ' for $\mathrm{Fix}(\psi)$ for a normal function ψ, $\psi'(\xi) = \psi(\xi)$ iff $\psi(\xi) = \xi$ iff $\psi'(\xi) = \xi$. Since γ is a successor ordinal, τ_ξ is ascending. Let $\chi = \sup_{\xi<D} \psi_k(\gamma, \pi_1, \alpha_{1\xi}, \ldots)$. By an argument as in the proof of theorem 5 $\chi \in \mathrm{Ran}(\zeta \mapsto \psi_k(\zeta, \pi_1, \alpha_1, \ldots)$. Since $\psi_k(\zeta^-, \pi_1, \alpha_1, \ldots) < \chi$ it suffices to show $\chi \leq \psi_k(\zeta, \pi_1, \alpha_1, \ldots)$, for which it suffices to show that $\psi_k(\gamma, \pi_1, \alpha_{1\xi}, \ldots) \leq \psi_k(\zeta, \pi_1, \alpha_1, \ldots)$ for all ξ. This follows because, letting $\delta = \psi_k(\zeta^-, \pi_1, \alpha_1, \ldots)$, $\psi_k(\delta, \pi_1, \alpha_{1\xi}, \ldots) = \psi_k(\zeta^-, \pi_1, \alpha_1, \ldots)$; since $\alpha_{1\xi} < \alpha_1$, $\psi_k(\delta + 1, \pi_1, \alpha_{1\xi}, \ldots) < \psi_k(\zeta^- + 1, \pi_1, \alpha_1, \ldots)$.

For case 3.0SL case 2 of the definition applies, and it suffices to show that $\pi \mapsto \psi_{k+1}(0, \pi, 1, \pi_1, \alpha_1^-, \ldots)$ is increasing. Again by case 2 of the definition, it suffices to show that $R_{S_2(\bar{\alpha},\xi+1)} \subseteq \mathrm{Fix}(R_{S_2(\bar{\alpha},\xi)})$. This follows using case 1 of the definition and facts from the proof of theorem 4.

For case 3.SSL, by an argument in case 3.0SL $R_{S_2(\bar{\alpha},\xi)}$ is strictly descending, whence by an argument as in case 3.SL* τ_ξ is ascending. Let $\chi = \sup_{\xi<D} \tau_\xi$. As in case 3.SL*, $\chi \in \mathrm{Ran}(\zeta \mapsto$

$\psi_k(\zeta, \pi_1, \alpha_1, \ldots))$, and in fact $\chi = \psi_k(\zeta, \pi_1, \alpha_1, \ldots)$.

Cases 3.0S0 and 3.SS0 follow because τ is respectively the first and next fixed point of $\zeta \mapsto \psi_k(\zeta, 0, \alpha_1^-, \ldots)$. Cases 3.0SS and 3.SSS follow because, by theorem 5.f, τ is respectively the first and next fixed point of $\xi \mapsto \psi_{k+1}(0, \pi_1^-, \xi, \pi_1, \alpha_1^-, \ldots)$. $\square$

5. Example 2: A scheme indexed by IV terms

A scheme Σ_{IVT} of length $\theta_{\mathrm{IV},\Omega^\varsigma}$ may be defined, by letting $\boldsymbol{\tau}$ range over normal IV terms in the base Ω, and using the ascending sequences defined above. The functions $\psi_{\boldsymbol{\tau}}$ for the terms $\boldsymbol{\tau}$ may be defined using $\triangle^<$ at stages of cofinality Ω. It is then an instance of a general fact that each $\psi_{\boldsymbol{\tau}}$ is normal.

It is clearly of interest whether the $R_{\boldsymbol{\tau}}$ are club subsets when the operation $\triangle^\leq$ is used at stages of cofinality Ω. Further discussion is omitted.

The closure ordinal $\theta_{\mathrm{IVT},\mu^\varsigma}$ of Σ_{IVT} is defined to be the smallest ordinal θ, such that if $\boldsymbol{\tau}$ is an IV term whose leaves are all less than θ, and $\zeta < \theta$, then $\psi_{\boldsymbol{\tau}}(\zeta) < \theta$. This is clearly an ordinal of interest. In the case $\mu = \omega$ it appears to be larger than the Bachmann-Howard ordinal; further discussion of this topic is omitted here.

6. Sequences of normal ultrafilters

A specific definition of a sequence of normal ultrafilters was given in [13], suitable for considering repeat points. A self-contained treatment will be given here, for convenience and with some improvements.

Let $o(U)$ denotes the Mitchell order of a normal ultrafilter U on a measurable cardinal κ, and $o(\kappa)$ the Mitchell order of κ. Define a sequence of normal ultrafilters on κ to be a function $\mathcal{U}_\kappa$ whose domain is some ordinal η, such that for $\beta < \eta$, $\mathcal{U}_\kappa(\beta)$ is a normal ultrafilter on κ, such that $o(\mathcal{U}_\kappa(\beta)) = \beta$. A sequence of normal ultrafilters is a function $\mathcal{U}$ whose domain is a set or class of measurable cardinals, such that for $\kappa \in \mathrm{Dom}(\mathcal{U})$, $\mathcal{U}(\kappa)$ is a sequence of normal ultrafilters on κ. $\mathcal{U}$ may be written indifferently for the unary predicate $\mathcal{U}(\langle\kappa, \beta, x\rangle)$, which holds iff $x \in \mathcal{U}(\kappa)(\beta)$ in the function notation; this is often done when defining Mitchell's model $L[\mathcal{U}]$.

Repeat points may be defined for a sequence of normal ultrafilters on κ. Say that S is a separating set for $\mathcal{U}_\kappa$ at α if $S \in \mathcal{U}_\kappa(\alpha)$ but $S \notin \mathcal{U}_\kappa(\beta)$ for $\beta < \alpha$; such exists iff α is not a repeat point.

Suppose $\mathcal{U}$ is a sequence of normal ultrafilters, $\kappa \in \mathrm{Dom}(\mathcal{U})$, and $0 \leq \beta < \mathrm{Dom}(\mathcal{U}(\kappa))$. In these circumstances U_β, or just U, will be used to denote $\mathcal{U}(\kappa)(\beta)$. Let M denote $\mathrm{Ult}_U(V)$, and let j denote the canonical embedding $j : V \mapsto M$. It is standard to define $j(\mathcal{U})$ to be the predicate on M, where if f_μ represents μ, f_γ represents γ, and f_x represents x, then $j(\mathcal{U})(\langle [f_\mu], [f_\gamma], [f_x] \rangle)$ iff $\{\lambda : \mathcal{U}(\langle f_\mu(\lambda), f_\gamma(\lambda), f_x(\lambda) \rangle)\} \in U$ (where λ ranges over cardinal less than κ).

Let $L_\in^{\mathcal{U}}$ be $L_\in$, expanded with a predicate symbol for $\mathcal{U}$. M may be expanded by interpreting the $\mathcal{U}$ symbol as $j(\mathcal{U})$.

Lemma 8. *(Los' theorem.) Suppose ϕ is a formula of $L_\in^{\mathcal{U}}$ and $[f_1], \ldots, [f_k]$ are elements of M. Then $\models_M \phi([f_1], \ldots, [f_k])$ iff $\{\lambda : \phi(f_1(\lambda), \ldots, f_k(\lambda))\} \in U$.*

Proof. This follows by the usual induction, since it holds for the $\mathcal{U}$ predicate by definition. □

Write $\langle \mu, \gamma \rangle < \langle \kappa, \beta \rangle$ for the lexicographic order, which holds iff $\mu < \kappa$ or $\mu = \kappa$ and $\gamma < \beta$. Let $\mathcal{U} \restriction \langle \kappa, \beta \rangle$ denote $\mathcal{U}$, restricted to those pairs $\langle \mu, \gamma \rangle$ such that $\langle \mu, \gamma \rangle < \langle \kappa, \beta \rangle$. Let $j(\mathcal{U}) \restriction \kappa + 1$ denote $j(\mathcal{U})$, restricted to those pairs $\langle \mu, \gamma \rangle$ such that $\mu \leq \kappa$. Say that a sequence of normal ultrafilters $\mathcal{U}$ is coherent if for all $\kappa \in \mathrm{Dom}(\mathcal{U})$ and $\beta < \mathrm{Dom}(\mathcal{U}(\kappa))$, $j(\mathcal{U}) \restriction \kappa + 1 = \mathcal{U} \restriction \langle \kappa, \beta \rangle$. Fix κ, β, etc. as above.

Suppose $f : \kappa \mapsto \mathrm{Ord}$. Let $D_f^{\geq} = \{\lambda : \mathrm{Dom}(\mathcal{U}(\lambda)) \geq f(\lambda\}$. By Los' theorem, $\mathrm{Dom}(\mathcal{U}(\kappa))^M \geq [f]$ iff $D_f^{\geq} \in U$. If $\mathcal{U}$ is coherent then $\beta \geq [f]$ iff $D_f^{\geq} \in U$.

For an interval $I \subseteq [0, \mathrm{Dom}(\mathcal{U}(\kappa)))$ say that a function f representing x on I if $[f]_{\mathcal{U}_\beta} = x$ for $\beta \in I$. Say that x is represented on I if there is some f representing x on I. For ease of notation ∞ may be written for $\mathrm{Dom}(\mathcal{U}(\kappa))$ as a right endpoint of an interval.

Suppose f represents α in I. If $\mathcal{U}$ is coherent then for $\beta \in I$, $\beta \geq \alpha$ iff $D_f^{\geq} \in U_\beta$.

7. T-separating sets

T-separating sets were defined in [14]. Again, a self-contained treatment will be given here. Suppose $\mathcal{U}$ is a sequence of normal ultrafilters. Say that S is a T-separating set for $\mathcal{U}(\kappa)$ at α if $S \in \mathcal{U}(\kappa)(\beta)$ if $\beta \geq \alpha$ but $S \notin \mathcal{U}(\kappa)(\beta)$ for $\beta < \alpha$.

Lemma 9. *If there is a separating set at α then there is a set S^G_α such that $S^G_\alpha \in U_\beta$ iff $\beta > \alpha$, and a function f^G_α such that f^G_α represents β mod U_β if $\beta \leq \alpha$, else α if $\beta > \alpha$.*

Proof. Let S be the separating set at α. Let $S^G_\alpha = \{\lambda : \exists\eta < \mathrm{Dom}(\mathcal{U}(\lambda))(S \cap \lambda \in \mathcal{U}(\lambda)(\eta))\}$. Then $S^G_\alpha \in U_\beta$ iff $\models_M \exists\eta < \mathrm{Dom}(\mathcal{U}(\kappa))(S \in \mathcal{U}(\kappa)(\eta))$. By coherence this holds iff $\exists\eta < \beta(S \in \mathcal{U}(\kappa)(\eta))$. Since S is separating at α this holds iff $\beta > \alpha$.

Let $f^G_\alpha(\lambda) = \sup_{\eta < \mathrm{Dom}(\mathcal{U}(\lambda))} \forall\eta' < \eta(S \cap \lambda \notin \mathcal{U}(\lambda)(\eta'))$. Using coherence, $[f^G_\alpha]_{U_\beta} = \sup_{\eta<\mathrm{Dom}(\mathcal{U}(\kappa))} \forall\eta' < \eta(S \notin \mathcal{U}(\kappa)(\eta'))$. The claim follows since S is a separating set. □

Lemma 10. *If there is a separating set at α and $\alpha+1 < Dom(\mathcal{U}(\kappa))$ then there is a T-separating set at $\alpha + 1$.*

Proof. By the hypothesis $\alpha + 1 < \mathrm{Dom}(\mathcal{U}(\kappa))$, S^G_α is a T-separating set at $\alpha + 1$. □

Theorem 11. *Suppose $\eta < \kappa$, for $\xi < \eta$ S_ξ is a T-separating set at α_ξ, $\alpha = \sup_{\xi<\eta} \alpha_\xi$, and $\alpha < Dom(\mathcal{U}(\kappa))$. Let $S = \cap_{\xi<\eta} S_\xi$; then S is a T-separating set at α.*

Proof. For $\beta \geq \alpha$, since $S_\xi \in U_\beta$ for $\xi < \eta$ and U_β is κ-complete, $S \in U_\beta$. If $\beta < \alpha$ then $\beta < \alpha_\xi$ for some ξ, so $S_\xi \notin U_\beta$, so $S \notin U_\beta$ since $S \subseteq S_\xi$. □

Theorem 12. *Suppose for $\xi < \kappa$ S_ξ is a T-separating set at α_ξ, $\alpha = \sup_{\xi<\kappa} \alpha_\xi$, and $\alpha < Dom(\mathcal{U}(\kappa))$. Let $S = \triangle_{\xi<\kappa} S_\xi$; then S is a T-separating set at α.*

Proof. For $\beta \geq \alpha$, since $S_\xi \in U_\beta$ for $\xi < \kappa$ and U_β is normal, $S \in U_\beta$. If $\beta < \alpha$ then $\beta < \alpha_\xi$ for some ξ, so $S_\xi \notin U_\beta$, so $S \notin U_\beta$ since $S \subseteq_{\mathcal{I}} S_\xi$ where $\mathcal{I}$ is the thin ideal. □

8. $\check{C}$

Suppose $\mathcal{U}$ is a coherent sequence of normal ultrafilters and $\kappa \in \mathrm{Dom}(\mathcal{U})$. Let C denote the function $\beta \mapsto j_{\mathcal{U}(\kappa)(\beta)}(\kappa)$ on $\mathrm{Dom}(\mathcal{U}(\kappa))$. More generally let C_λ denote the function $\beta \mapsto j_{\mathcal{U}(\lambda)(\beta)}(\lambda)$.

Lemma 13. *Suppose κ is a measurable cardinal. If $U \triangleleft W$ then $j_U(\kappa) < j_W(\kappa)$.*

Proof. Suppose U is represented mod W by $\lambda \mapsto U_\lambda$. Then $j_U(\kappa)$ is represented by $\lambda \mapsto j_{U_\lambda}(\lambda)$ and $j_{U_\lambda}(\lambda) < \kappa$. □

In [13] it is claimed that C is continuous, but the proof given is incorrect and in fact the claim is false.

Lemma 14. *a. If $\alpha < \beta$ then $C(\alpha)^M = C(\alpha)$.*
b. If α is a limit ordinal and $\alpha \leq \beta$ then $\check{C}(\alpha)^M = \check{C}(\alpha)$.

Proof. For part a, $C(\alpha)$ is the rank of a well-preorder on κ^κ. κ^κ is the same in M and V, and $\mathcal{U}(\kappa)(\alpha)$ also since $\alpha < \beta$ and $\mathcal{U}$ is coherent. Thus, the well-preorder is the same, and the rank function is Δ_1 in $L_\in$. For part b, by part a $C(\zeta)^M = C(\zeta)$ for $\zeta < \alpha$ and the claim follows. □

Theorem 15. *For any limit ordinal $\xi <$ Dom$(\mathcal{U}(\kappa))$, $\sup_{\zeta<\xi} C(\zeta) < C(\xi)$.*

Proof. Let $U = \mathcal{U}(\kappa)(\xi)$ and $M = \mathrm{Ult}_U(V)$. For $\zeta < \xi$ let f_ζ represent ζ mod U. The function $g_\zeta(\lambda) = C_\lambda(f_\zeta(\lambda))$ represents $C(\zeta)$ and satisfies $g_\zeta(\lambda) < (2^\lambda)^+$ for all λ. It follows that $C(\zeta)^M < ((2^\kappa)^+)^M < C(\xi)$ and the theorem follows using lemma 14. □

There is a useful modified version of C. Let c be an increasing function from some ordinal to Ord. Let $\check{c}(\xi)$ equal $c(\xi)$ if ξ is a successor ordinal, else $\sup_{\zeta<\xi} c(\zeta)$.

Lemma 16. *$\check{c}$ is an increasing and continuous function with the same domain as c. $\check{c}(\xi) \leq c(\xi)$, and equality holds for successor ordinals.*

Proof. Left to the reader. □

Lemma 17. *If there is a separating set at α and $\nu \in (\alpha, \min($Dom $(\mathcal{U}(\kappa)), C(\alpha)))$ then there is a T-separating set at ν.*

Proof. Since $\nu < C(\alpha)$ there is a function $g : \kappa \mapsto \kappa$ representing ν mod U_α. For $\lambda \in S^G_\alpha$ let $\tilde{g}(\lambda) = [g \restriction \lambda]_{\mathcal{U}(\lambda)(f_\alpha(\lambda))}$; otherwise let $\tilde{g}(\lambda) = 0$.

Suppose $\beta > \alpha$ and let U, M, j be as usual. Then in M, $\tilde{g}$ represents $[j(g) \restriction \kappa]_{j(\mathcal{U})(\kappa)(\alpha)}$. But $j(g) \restriction \kappa$ equals g and by coherence $j(\mathcal{U})(\kappa)(\alpha) = U_\alpha$, and so $\tilde{g}$ represents ν. □

Lemma 18. *Suppose f is normal and $\theta \in Lim$. Then $\alpha < \theta \Rightarrow f(\alpha) < \theta$ iff $f(\theta) = \theta$.*

Proof. $\theta \in [f(\alpha), f(\alpha+1))$ for a unique α. If $f(\alpha) < \theta$ then $\alpha < \theta$ so $\alpha+1 < \theta$ so $f(\alpha+1) < \theta$, a contradiction. Thus, $\theta = f(\alpha)$. If $\alpha < \theta$ then again $\alpha + 1 < \theta$ so $f(\alpha + 1) < \theta$. Thus, $\alpha = \theta$. For the converse, if $\zeta < \theta$ then $f(\zeta) < f(\theta) = \theta$. □

If there is an ordinal $\theta < \mathrm{Dom}(\mathcal{U}(\kappa))$ such that there is no T-separating set at θ let θ_T be the smallest such.

Theorem 19. *θ_T is a limit ordinal. $\theta_T = \check{C}(\theta_T)$. Cf$(\theta_T) \geq \kappa^+$ (if GCH holds then Cf$(\theta_T) = \kappa^+$).*

Proof. If $\alpha < \theta_T$ then $\alpha + 1 \leq \theta_T < \mathrm{Dom}(\mathcal{U}(\kappa))$ so by lemma 10 $\alpha + 1 < \theta_T$; thus, θ_T is a limit ordinal. If $\alpha < \theta_T$ then using lemma 17 inductively, if $\alpha \leq \nu < C(\alpha)$ then $\nu < \theta_T$. Further, since $\alpha < \alpha + 1 < \theta_T$, $C(\alpha) < C(\alpha + 1)$ and so $C(\alpha) < \theta_T$. (This also follows using lemma 14). $\theta_T = \check{C}(\theta_T)$ now follows by lemma 18. That $\mathrm{Cf}(\theta_T) \geq \kappa^+$ follows by theorems 11 and 12. □

9. $\theta_{\mathbf{IVT},\check{C}} \leq \theta_T$

In this section it will be assumed that GCH holds in V. Suppose κ is a measurable cardinal. Let $\Omega = \kappa^{++}$, and let Σ_{IVT} be as in section 5. Let $\psi_0 = \check{C}$, and for $\tau \neq 0$ let $\psi_{\boldsymbol{\tau}}$ be the function determined (using Fix at successor stages and $\triangle^<$ at stages of cofinality Ω) according to Σ_{IVT}. Note that $\mathrm{Dom}(\psi_{\boldsymbol{\tau}}) \leq \mathrm{Dom}(\mathcal{U}(\kappa))$, and may in fact be $\emptyset$. Note also that by lemma 3, for $\tau \neq 0$ any nonzero element of $R_{\boldsymbol{\tau}}$ is a fixed point of $\check{C}$.

Let $\theta_{\mathrm{IVT},\check{C}}$ be the closure ordinal, that is, the smallest ordinal $\theta > 0$ such that if $\boldsymbol{\tau}$ is an IV term whose leaves are all less than θ, and $\zeta < \theta$, then $\psi_{\boldsymbol{\tau}}(\zeta) < \theta$. A term $\boldsymbol{\tau}$ may be written as $\boldsymbol{\tau}(\alpha_1, \ldots, \alpha_k)$ to indicate that the leaves are the ordinals $\alpha_1, \ldots, \alpha_k$.

Theorem 20. *Suppose there are T-separating sets at each of* $\alpha_1, \ldots, \alpha_k, \zeta < Dom(\mathcal{U}(\kappa))$*, and* $\zeta \in Dom(\psi_{\boldsymbol{\tau}(\alpha_1,\ldots,\alpha_k)})$*. Then there is a T-separating set at* $\psi_{\boldsymbol{\tau}(\alpha_1,\ldots,\alpha_k)}(\zeta)$*.*

Proof. For $o = \alpha_1, \ldots, \alpha_k, \zeta$ let S_o be a T-separating set at o. Write $\psi_{\boldsymbol{\tau}}$ for $\psi_{\boldsymbol{\tau}(\alpha_1,\ldots,\alpha_k)}$. Let Meas denote the measurable cardinals. For $\lambda \in \text{Meas} \cap \kappa$ write ψ_{τ_λ} for the function $\psi_{\boldsymbol{\tau}(\psi^G_{\alpha_1}(\lambda),\ldots,f^G_{\alpha_k}(\lambda))}$ where $\boldsymbol{\tau}$ is interpreted at λ. Let $S_1 = \{\lambda \in \text{Meas} : f^G_\zeta(\lambda) \in \text{Dom}(\psi_{\tau_\lambda})\}$. Let $S_2 = \{\lambda \in S_1 : Dom(\mathcal{U}(\lambda)) \geq \psi_{\tau_\lambda}(f^G_\zeta(\lambda))\}$. Let $S = S_{\alpha_1} \cap \cdots \cap S_{\alpha_k} \cap S_\zeta \cap S_2$.

Suppose $\beta < \max(\alpha_1, \cdots, \alpha_k, \zeta)$. Then $S \notin U$.

Suppose $\max(\alpha_1, \cdots, \alpha_k, \zeta) \leq \beta < \psi_{\boldsymbol{\tau}}(\zeta)$. Now, if $\zeta \in \text{Dom}(\psi_{\boldsymbol{\tau}})$ then $\psi_{\boldsymbol{\tau}}(\zeta)$ is 0 or a fixed point of $\check{C}$, and so $\psi_{\boldsymbol{\tau}}(\zeta) \leq \text{Dom}(\mathcal{U}(\kappa))$. Since this holds in M, in M $\zeta \notin \text{Dom}(\psi_{\boldsymbol{\tau}})$, and so $S \notin U$.

Suppose $\beta \geq \psi_{\boldsymbol{\tau}}(\zeta)$. By hypothesis $\alpha \in \text{Dom}(\psi_{\boldsymbol{\tau}})$, so by lemma 14.b $\alpha \in \text{Dom}(\psi_{\boldsymbol{\tau}})^M$; also $\beta \geq \psi_{\boldsymbol{\tau}}(\beta)^M$. Thus, $S \in U$.

Thus, S is a T-separating set at $\psi_{\boldsymbol{\tau}(\alpha_1,\ldots,\alpha_k)}(\zeta)$. ☐

Theorem 21. *Suppose* $\alpha_1, \ldots, \alpha_k < \theta_T$*. Then* $\psi_{\boldsymbol{\tau}(\alpha_1,\ldots,\alpha_k)}(\theta_T) = \theta_T$*.*

Proof. If $\theta_T = \psi_{\boldsymbol{\tau}(\alpha_1,\ldots,\alpha_k)}(\zeta)$ then by theorem 20 ζ must equal θ_T. Thus it suffices to show that $\theta_T \in R_{\boldsymbol{\tau}(\alpha_1,\ldots,\alpha_k)}$. This may be proved by induction on $\boldsymbol{\tau}$. The basis $\tau = 0$ follows by theorem 19. If $\tau = \tau^- + 1$, then inductively $\theta_T \in R_{\tau^-}$, and by what has just been shown $\theta_T \in \text{Fix}(R_{\tau^-}) = R_\tau$.

An examination of the ascending sequences of IV terms given in section 4 yields that if $R_\tau = \cap_{\xi<\eta} R_{\tau_\xi}$ then the leaves of each τ_ξ are less than θ_T, whence inductively $\theta_T \in R_{\tau_\xi}$ for all ξ, whence $\theta_T \in R_\tau$. If $R_\tau = \triangle R_{\tau_\xi}$ one also sees that the leaves of each τ_ξ for $\xi < \theta_T$ are less than θ_T, whence inductively $\theta_T \in R_{\tau_\xi}$ for all $\xi < \theta_T$, whence $\theta_T \in R_\tau$. ☐

Corollary 22. $\theta_{IVT,\check{C}} \leq \theta_T$.

Proof. Straightforward. ☐

10. Mitchell's model

The standard definition of a coherent sequence in V (as found in [20] for example) does not require $o(\mathcal{U}(\kappa)(\beta)) = \beta$. As seen above, this restriction is useful for considering repeat points.

The coherent sequence $\mathcal{U}$ of Mitchell's model $L[\mathcal{U}]$ (see [19]) is of particular interest. In this case the coherent sequence has further properties, including the following, as may readily be verified from the results of [19].

- $o(\mathcal{U}(\kappa)(\beta)) = \beta$
- Every ultrafilter of $L[\mathcal{U}]$ is $\mathcal{U}(\kappa)(\beta)$ for some κ, β.
- The coherence requirement $j(\mathcal{U}) \restriction \kappa + 1 = \mathcal{U} \restriction \langle \kappa, \beta \rangle$ holds.
- Since $\mathcal{U}$ enumerates the normal ultrafilters and j is elementary $j(\mathcal{U})$ enumerates the normal ultrafilters of M. Further, $M = L[j(\mathcal{U})]$.

11. An absoluteness principle

In this section some facts will be proved, which will be used below and may be useful in further research. Fix κ, β, etc. as usual.

Lemma 23. *Suppose ϕ is a Δ_0 sentence of $L^{\mathcal{U}}_{\in}$ whose parameters are all in $L_\beta[\mathcal{U}]$. Then $\models_M \phi$ iff ϕ.*

Proof. The proof is by induction on ϕ. For the $\in$ predicate the claim follows since M is a $\in$-substructure of V. For the $\mathcal{U}$ predicate it follows by coherence. For the propositional connectives it follows by standard arguments. If ϕ is $\exists x \in b\psi$ then b is a parameter and the claim follows inductively as usual. $\square$

Corollary 24. *Suppose ϕ is a Δ_1 sentence of $L^{\mathcal{U}}_{\in}$ whose parameters are all in $L_\beta[\mathcal{U}]$. Then $\models_M \phi$ iff ϕ.*

Proof. This follows from lemma 23 as usual. $\square$

The next lemma gives a useful fact about $L[A]$.

Lemma 25. *There is a definable surjection $F : Ord \mapsto L[A]$ such that if θ is an ϵ-number then $L_\theta[A] = F[\theta]$. Further F is definable by a Δ_1 formula which defines the predicate $x = F(\alpha)$, uniformly in $L_\theta[A]$ for limit ordinals $\theta > \omega$ and uniformly in A.*

Proof. The "main theorem" of [17] states that $L_\alpha \in F(\omega^{\omega^{\alpha\cdot 2+1}})$ if α is infinite, and $L_\alpha \in F(\omega)$ for finite α, where F is a certain definable function defined by Godel. The relativized version replaces L_α by $L_\alpha[A]$ for a class A, and F with a modified version which is definable from A. The operation $x \mapsto x \cap A$ is added to the list of fundamental operations. The relativized version may be proved by modifying the proof of the main theorem given in [17], treating $x \cap A$ similarly to $x \cap \epsilon$.

It is routine to verify that for limit ordinals $\theta > \omega$ $L_\theta[A]$ is closed under the fundamental operations. Further there is a Δ_1 formula which defines the predicate $s = F \restriction \alpha$, uniformly in $L_\theta[A]$ and A; and using this such a formula for $x = F(\alpha)$ may be defined. See exercise II.7.2 of [3] and [18] for various details. $\square$

The next theorem is a generalization of a remark following proposition 3.9 in [26]. Suppose $V = L[\mathcal{U}]$. Noting that $\mathcal{U}$ is a definable class in $L[\mathcal{U}]$, let $\mathcal{F}_{\Delta_1^{ZFC}} = \{f : V \mapsto V :$ f is defined by a Σ_1 formula ϕ of $L^{\mathcal{U}}_{\in}\}$ and $\forall x \exists! y \phi(x, y)$ is provable in ZFC.

Theorem 26. *In $L[\mathcal{U}]$, suppose $\theta \leq Dom(\mathcal{U}(\kappa))$ and $\theta = \check{C}(\xi)$ for some limit ordinal ξ. Then for $f \in \mathcal{F}_{\Delta_1^{ZFC}}$, $f[L_\theta[\mathcal{U}]] \subseteq L_\theta[\mathcal{U}]$.*

Proof. Suppose first that $f : \mathrm{Ord} \mapsto \mathrm{Ord}$ and $\alpha < \theta$. Choose $\xi_0 < \xi$ with $\alpha < C(\xi_0)$ and let $\theta_0 = C(\xi_0)$. Let $U = \mathcal{U}(\kappa)(\theta_0)$ and $M = \mathrm{Ult}_U(V)$. Then θ_0 is inaccessible in M. Since M is a model of ZFC $f(\alpha)^M < \theta_0$. By lemma 24 $f(\alpha)^V = f(\alpha)^M$. The claim is proved for $f : \mathrm{Ord} \mapsto \mathrm{Ord}$.

The preceding argument works equally well for $f : \mathrm{Ord}^2 \mapsto \mathrm{Ord}$. In particular θ is closed under ordinal exponentiation and so is an ϵ-number. Let F be the function of lemma 25. Using exercise II.7.2.L of [3], there is a right inverse G to F, which is uniformly Δ_1 in $L_\theta[A]$ for ϵ-numbers θ. Given $f \in \mathcal{F}_{\Delta_1^{ZFC}}$ let $f' = GfF$, so that $f' : \mathrm{Ord} \mapsto \mathrm{Ord}$, whence by what has already been proved $f'[\theta] \subseteq \theta$. The claim follows. $\square$

Referring to [15], the primitive recursive ordinal functions are in $\mathcal{F}_{\Delta_1^{ZFC}}$, and so θ as in the theorem is Prim_O-closed.

Lemma 27. *Suppose $V = L[\mathcal{U}]$ and $0 \leq \theta < \beta < \kappa^{++}$; then $L_\theta[\mathcal{U}] \in M$. Further, a representing function for θ may be transformed to a representing function for $L_\theta[\mathcal{U}]$, with the result depending only on f_θ and not β.*

Proof. This follows using exercise II.7.2.G of [3] and corollary 24. □

Theorem 28. *Suppose* $V = L[\mathcal{U}]$ *and* $\theta \in Ord$. *Then every* $x \in L_\theta[\mathcal{U}]$ *is represented on* $[\theta, \infty)$.

Proof. It will be shown by induction on $\rho \leq \theta$ that every $x \in L_\rho[\mathcal{U}]$ is represented on $[\theta, \infty)$. The basis $\rho = 0$ is trivial. Suppose $x \in L_{\rho+1}[\mathcal{U}] - L_\rho[\mathcal{U}]$. Suppose $x = \{w \in L_\rho[\mathcal{U}] : \models_{L_\rho[\mathcal{U}]} \phi(w, \vec{p})\}$ where $p_1, \ldots, p_k \in L_\rho[\mathcal{U}]$. Inductively, there are $f_\rho, f_{p_1}, \ldots, f_{p_k}$ representing $\rho, p_1, \ldots, p_k$ in $[\theta, \infty)$. Define f_x by $f_x(\lambda) = \{w \in L_{f_\rho(\lambda)}[\mathcal{U}] : \models_{L_{f_\rho(\lambda)}[\mathcal{U}]} \phi(w, f_{p_1}(\lambda), \ldots, f_{p_k}(\lambda))\}$. Suppose $w \in L_\rho[\mathcal{U}]$ and let f_w be the recursively constructed function representing w in $[\theta, \infty)$. Suppose $\beta \in [\theta, \infty)$ and let U, etc. be as usual. Since satisfaction is Δ_1, by lemmas 27 and 23, $w \in x$ iff $(w \in x)^M$. The claim for $\rho \in \mathrm{Lim}$ follows inductively. □

In particular an ordinal ρ is represented in (ρ, ∞). It is a question of interest whether it is represented in $[\rho, \infty)$.

12. O-schemes in $L[\mathcal{U}]$

In this section it will be assumed that $V = L[\mathcal{U}]$. By an O-scheme over a regular uncountable cardinal Ω is meant a pair $\langle \prec, \phi \rangle$ where $\prec$ is a well-order on a subset of $L_\Omega[\mathcal{U}]$ and ϕ is a function satisfying the same requirements as for a scheme, when $\mathrm{Fld}(\prec)$ is identified with the order type of $\prec$. Indeed, an O-scheme Σ may be identified with a scheme in this way, and $F^{\Sigma^o, x}(X)$ for $x \in \mathrm{Fld}(\prec)$ may be defined via this identification. This set will also be denoted as R_x, and $\mathrm{Enum}(R_x)$ as ψ_x.

Note that there is an ordinal ρ such w.l.g. Σ can be required to be in $L_\rho[\mathcal{U}]$. It is a question of interest what bounds on ρ can be given.

From hereon X will be taken as $\mathrm{Ran}(\check{C})$ and F as Fix. Let Σ be an O-scheme. Let x_0 be the least element under $\prec$. The closure ordinal θ_Σ is defined to be the smallest θ such that $x_0 \in L_\theta[\mathcal{U}]$, and if $x \in L_\theta[\mathcal{U}]$ and $\zeta < \theta$ then $\psi_x(\zeta) < \theta$.

Theorem 29. *Suppose there is a class* $\mathcal{S}$ *of O-schemes with the following two properties.*

1. For $\Sigma \in \mathcal{S}$ $\theta_\Sigma \leq \theta_T$.

2. For any $\theta < \kappa^{++}$ there is a $\Sigma \in \mathcal{S}$ such that $\theta_\Sigma \geq \theta$.

Then repeat points do not exist in $L[\mathcal{U}]$.

Proof. Immediate. □

The rest of this section gives a discussion of questions relevant to theorem 29.

By lemma 3, for $x \neq x_0$ any nonzero element of R_x is a fixed point of $\check{C}$.

Say that an O-scheme Σ is T-extending if whenever there is a T-separating set at α, $x \in \mathrm{Fld}(\prec) \cap L_\alpha[\mathcal{U}]$, and there is a T-separating set at ζ, then there is a T-separating set at $\psi_x(\zeta)$.

Say that an O-scheme Σ is tempered if whenever θ is Prim_O-closed and $x \in L_\theta[\mathcal{U}]$ the following hold: If $R_x = \cap_{\xi<\eta} R_{x_\xi}$ then $x_\xi \in L_\theta[\mathcal{U}]$. If $R_x = \triangle_\xi R_{x_\xi}$ then $x_\xi \in L_\theta[\mathcal{U}]$ for $\xi < \theta$.

Say that x_0 is suitable if $x_0 \in L_{\theta_T}[\mathcal{U}]$.

Theorem 30. *If Σ is a T-extending and tempered O-scheme with suitable x_0 then $\theta_\Sigma \leq \theta_T$.*

Proof. The proof of corollary 22 is readily modified. □

13. $W(\xi)$

In this section it will be assumed that the universe is Mitchell's model $L[\mathcal{U}]$, with κ, β, U, M, j as usual. Recall the definition of a WPS as given in [13]; as in that reference, in this section let Ω denote the rank function.

A function $W : \mathrm{Ord} \mapsto \mathrm{Ord}$ will be define in $L[A]$ for a class A, uniformly in A. Given $\theta \in \mathrm{Ord}$ let $\mathrm{W}^{\preceq}_{\mathrm{nxt}}(\theta) = \{\preceq : \preceq$ is a WPS definable in $L_\theta[A]$ from parameters$\}$. Let $\mathrm{W}_{\mathrm{nxt}}(\theta) = \Omega[\mathrm{W}^{\preceq}_{\mathrm{nxt}}(\theta)]$. It is readily verified that $\mathrm{W}_{\mathrm{nxt}}(\theta)$ is a limit ordinal and $\mathrm{W}_{\mathrm{nxt}}(\theta) > \theta$. Given a parameter ρ $W^\rho(\xi)$ is defined by recursion on ξ as follows. $W^\rho(0) = \rho$. $W^\rho(\xi+1) = \mathrm{W}_{\mathrm{nxt}}(W^\rho(\xi))$. If ξ is a limit ordinal then $W^\rho(\xi) = \cup_{\zeta<\xi} W^\rho(\zeta)$. In the case of interest, $\rho = \kappa^+$ for some $\kappa \in \mathrm{Dom}(\mathcal{U})$ and W may be written rather than W^{κ^+}.

Lemma 31. *$W_{nxt}(\theta)$ is an ϵ-number. If ρ is an ϵ-number then for all ξ $W^\rho(\xi)$ is an ϵ-number.*

Proof. It suffices to show that $\hat{\theta} = \mathrm{W}_{\mathrm{nxt}}(\theta)$ is closed under $\langle \alpha, \beta \rangle \mapsto \alpha^{\beta}$. Let $\alpha = \Omega(\preceq_A)$, and let $\beta = \Omega(\preceq_B)$. Let C be the relation defined in L_θ as follows. $\mathrm{Fld}(C)$ equals the set of functions with domain an integer n, where for $i < n$ $f(i) \in \mathrm{Fld}(\preceq_B) \times \mathrm{Fld}(\preceq_A)$, such that if $f(i) = \langle b_1, a_1 \rangle$ and $f(i+1) = \langle b_2, a_2 \rangle$ then $b_1 \prec_B b_2$. Given $a_1, a_2 \in \mathrm{Fld}(A)$ and $b_1, b_2 \in \mathrm{Fld}(B)$ write $\langle b_1, a_1 \rangle \sim \langle b_2, a_2 \rangle$ if $b_1 \equiv_B b_2$ and $a_1 \equiv_A a_2$. Given $f_1, f_2 \in \mathrm{Fld}(\preceq_C)$ say that $f_1 \preceq_C f_2$ if either $\mathrm{Dom}(f_1) \leq \mathrm{Dom}(f_2)$ and $f_1(i) \sim f_2(i)$ for all $i < \mathrm{Dom}(f_1)$; or if j is the "first difference", $f_1(j) = \langle b_1, a_1 \rangle$, and $f_2(j) = \langle b_2, a_2 \rangle$, then either $b_1 \prec_B b_2$ or $b_1 \equiv_B b_2$ and $a_1 \prec_A a_2$. It is readily seen that C is a WPS, and using well-known facts about ordinal arithmetic (see section 16 of [23] for example) $\Omega(C) = \alpha^{\beta}$. That $W^{\rho}(\xi)$ is an ϵ-number follows by induction on ξ. $\square$

As a fact of interest, note that if θ is an ϵ-number then $\mathrm{W}^{\preceq}_{\mathrm{nxt}}(\theta)$ equals $\{\preceq : \preceq$ is a WPS definable in $L_\theta[A]$ from ordinal parameters$\}$. This follows using lemma 25, because a definition for a WPS in $L_\theta[A]$ may be transformed to a definition from ordinal parameters using a definition of F.

Lemma 32. *If $\theta' = W_{nxt}(\theta)$ then θ' is definable in $L_{\theta+1}[A]$.*

Proof. Let $P(x)$ be the predicate, "x is a binary relation on $L_\theta[A]$ which is a WPS". Let $Q(x, y)$ be the predicate, "$P(x)$ and $P(y)$ and there is an order preserving injection from x to y. Q defines a WPS of order type θ'. $\square$

Lemma 33. *The function $W^{\rho}(\xi)$ is Δ_1-definable, uniformly in A and ρ.*

Proof. The predicate $\theta_2 = \mathrm{W}_{\mathrm{nxt}}(\theta_1)$ may be defined by the following formula: $\forall \alpha < \theta_2 \exists r \in L_{\theta_1+1}[A](\mathrm{WPS}(r) \wedge \alpha = \Omega(r) \wedge \forall r \in L_{\theta_1+1}[A](\mathrm{WPS}(r) \Rightarrow \Omega(r) < \theta_2)$. Using well-known facts this formula may be seen to be Δ_1. The formulas $s = W^{\rho} \upharpoonright \xi$ and $\theta = W^{\rho}(\xi)$ may then be seen to be Δ_1. $\square$

14. W and θ_T

In this section $V = L[\mathcal{U}]$ will be assumed.

Theorem 34. *Suppose $\theta = \check{C}(\xi)$ for some limit ordinal ξ and $\theta < Dom(\mathcal{U}(\kappa))$. Then $W(\theta) = \theta$.*

Proof. By theorem 26 an lemma 33 $W[\theta] \subseteq \theta$. The theorem follows since W is increasing and continuous. $\square$

In particular, $W(\theta_T) = \theta_T$. This is a fact of interest, but has has yet to yield any improvement to bounds on θ_T.

Lemma 35. *If $Pow(\kappa) \subseteq L_\theta[\mathcal{U}]$ and $\theta < Dom(\mathcal{U}(\kappa))$ then $C(\theta) < W_{nxt}(\theta)$.*

Proof. As noted in the proof of lemma 14.a, $C(\theta)$ is the rank of a well-preorder on κ^κ. Under the hypotheses, this preorder is definable over $L_\theta[\mathcal{U}]$. $\square$

Theorem 36. *Suppose $\theta < Dom(\mathcal{U}(\kappa))$ and $\theta = \check{C}(\xi)$ where ξ is a limit ordinal of uncountable cofinality. Then for $\theta_1 < \theta$, $Pow(\kappa) \not\subseteq L_{\theta_1}[\mathcal{U}]$.*

Proof. Suppose $\theta_2 = \sup_{i<\omega} C^i(\theta_1)$; then $\theta_2 < \theta < \mathrm{Dom}(\mathcal{U}(\kappa))$. Let ξ_i for $i > 0$ be such that $C^i(\theta_1) = C(\xi_i)$. Let $\xi = \sup_i \xi_i$. Then ξ is a limit ordinal and $\theta_2 = \check{C}(\xi)$. By lemma 35 $\theta_2 \leq \sup_i \mathrm{W}^i_{\mathrm{nxt}}(\theta_1)$. By theorem 26 $\sup_i \mathrm{W}^i_{\mathrm{nxt}}(\theta_1) < \theta_2$, a contradiction. $\square$

Corollary 37. *If $\theta < \theta_T$ then $Pow(\kappa) \not\subseteq L_\theta[\mathcal{U}]$.*

Proof. θ_T satisfies the hypotheses of theorem 36. $\square$

In view of this, and as a question in itself, it is of interest to obtain upper bounds on θ such that $\mathrm{Pow}(\kappa) \subseteq L_\theta[\mathcal{U}]$.

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