A VARIANT OF RECONSTRUCTIBILITY OF COLORED GRAPHS

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Abstract: A variant of reconstructibility of colored graphs is defined, and some facts proved. Some computations facts from an earlier paper are revised.

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1. Introduction

Colored graph reconstructibility has been considered since the early 1970's (see [1]). More recent references include [7],[6],[3]. Using terminology from [3], define a C-graph to be a graph, with colors assigned to the vertices and edges. C-graphs are called colored graphs in the literature. An isomorphism ϕ of C-graphs must preserve colors (i.e., for a vertex v $\phi(v)$ must have the same color as v and similarly for edges). A C-graph is defined to be reconstructible if it is determined by its deck. That is, if G and H have the same deck, in that the members of the two decks can be paired as isomorphic pairs, then G and H are isomorphic.

Given C-graphs G and H, define ϕ to be a $\bar{\mathrm{C}}$-isomorphism if $\phi(v_1)$ and $\phi(v_2)$ have the same color whenever v_1 and v_2 do; and similarly for edges. A C-graph is defined to be $\bar{\mathrm{C}}$-reconstructible if, whenever G and H have $\bar{\text{C}}$-isomorphic decks, in that the members of the two decks can be paired as $\bar{\text{C}}$-isomorphic pairs, then G and H are $\bar{\text{C}}$-isomorphic.

Sections 6 and 7 of [3] contain many errors due to confusion of C-reconstructibility $\bar{\text{C}}$-reconstructibility, and will be completely revised here.

All graphs will be assumed to have at least three vertices. For a graph G $V(G)$ denotes the vertices, $E(G)$ the edges, and for $v \in V(G)$ G_v denotes the point-deleted subgraph.

2. Basic facts

Define a $\bar{\text{C}}$-graph G to be a graph, together with partitions of its set of vertices and set of edges. An isomorphism between $\bar{\text{C}}$-graphs must preserve the partitions. If v is a vertex, G_v is the point deleted subgraph, together with the induced partitions, where two vertices or edges belong to the same part in G_v iff they do in G. To a C-graph G there corresponds a $\bar{\text{C}}$-graph $\bar{G}$, where a part is the vertices or edges of a given color.

Theorem 1. *Two C-graphs G, H are $\bar{C}$-isomorphic iff $\bar{G}, \bar{H}$ are isomorphic.*

Proof. Indeed, a bijection ϕ from the vertex set $V(G)$ to $V(H)$ is a $\bar{\text{C}}$-isomorphism from G to H iff it is an isomorphism from $\bar{G}$ to $\bar{H}$. □

Theorem 2. *A C-graph G is $\bar{C}$-reconstructible iff the corresponding $\bar{C}$-graph $\bar{G}$ is reconstructible.*

Proof. Suppose $\bar{G}$ is reconstructible and G, H have $\bar{\text{C}}$-isomorphic decks. By theorem 1 $\bar{G}, \bar{H}$ have isomorphic decks, whence by hypothesis $\bar{G}, \bar{H}$ are isomorphic, whence by theorem 1 G, H are $\bar{\text{C}}$-isomorphic. Suppose G is $\bar{\text{C}}$-reconstructible and $\bar{G}, \bar{H}$ have isomorphic decks. A similar argument shows that $\bar{G}, \bar{H}$ are isomorphic. □

As in [3] define a V-graph to be a graph, with colors assigned to the vertices (alternatively a C-graph with constant edge color); and an E-graph to be a graph with edge colors. Similarly a $\bar{\text{V}}$-graph (resp. $\bar{\text{E}}$-graph) is a graph with a vertex (resp. edge) partition.

The notion of Ē-reconstructibility is of little interest. Indeed, all three edge partitionings of K_3 have the same deck. There are 25 edge partitionings of K_4, having 11 decks. Hereafter, only V̄-graphs will be considered.

Theorem 3. *The multiset of part sizes of a $\bar{V}$-graph is reconstructible.*

Proof. Letting G denote the graph and n_v the number of vertices, the part size multiset of G is 1^{n_v} iff the part size multiset of each G_v is 1^{n_v-1}. Otherwise, the number of parts is the maximum such among the G_v. Let S be the lexicographically greatest part size multiset among the G_v; the part size multiset of G is readily obtained from S. □

Corollary 4. *For a vertex v in a $\bar{V}$-graph G, the size of the part containing v is known from G_v.*

Proof. This value is the largest size of a part of G, whose multiplicity is 1 less in G_v. □

Corollary 5. *A regular $\bar{V}$-graph G is reconstructible*

Proof. Let v be such that the part size of v is minimal. G may be reconstructed from G_v. □

Theorem 6. *A $\bar{V}$-graph G is reconstructible iff its complement G^c is.*

Proof. This follows because $(G^c)_v = (G_v)^c$. □

A basic fact about V-graphs is that a disconnected V-graph is reconstructible. Essentially the same argument (see theorem 3 of [3]) shows that for a V̄-graph G, the components together with their vertex partitions are reconstructible. However, it does not follow (at least readily) that G is reconstructible.

If G is a V-graph, G may be represented by a bipartite graph G_r which has a vertex class V for the vertices of G and a vertex class C for the colors. The edges of G_r are those of G, an an edge $\{v, c\}$ if v has color c. It is readily seen that given two V-graphs G, H with the same colors, G is isomorphic to H iff G_g and H_r are isomorphic by an isomorphism fixing V setwise and C pointwise;

and $\bar{G}$ is isomorphic to $\bar{H}$ iff G_g and H_r are isomorphic by an isomorphism fixing V and C setwise. This observation will be used in the computations below.

3. Computations for V-graphs

This section revises section 6 of [3].

Theorem 7. *For* $3 \leq |V(G)| \leq 9$*,* $\bar{G}$ *is reconstructible.*

Proof. For $|V(G)| = 3$ the 14 cases of $\bar{G}$ may be enumerated, and the decks seen to be distinct.

For $|V(G)| \geq 4$ the claim may be verified by a computer program. By results of [4] the underlying graph G is reconstructible. By theorem 6 only G where $|E(G)| \leq n(n-1)/4$ need be considered. By theorem 3, letting P denote the multiset of vertex partition part sizes, the $\bar{\text{V}}$-graphs for each G and P may be considered separately. Representing them as noted above, the $\bar{\text{V}}$-graphs may be canonicalized up to setwise fixing of the partition parts using the Nauty [5] library. Reconstructibility may be verified by canonicalizing the decks, and verifying that distinct canonicalized $\bar{\text{V}}$-graphs have distinct canonicalized decks. □

Theorem 8. *For* $3 \leq |V(G)| \leq 9$*,* G *is V-reconstructible.*

Proof. By theorem 7, the V-graphs with a given $\bar{\text{V}}$-graph may be considered separately. In a vertex coloring, two parts may not have their colors exchanged if (A) they have different sizes, or (B) they have different degree sequences.

For $|V(G)| = 3$, for 6 of 14 $\bar{\text{V}}$-graphs there is a single isomorphism class of vertex colorings, for 6 of them there are two classes which may be distinguished by criterion (A), and for 2 of them there are three classes which may be distinguished by criterion (B).

For $|V(G)| \geq 4$ the claim may be verified by a computer program. The $\bar{\text{V}}$-graphs may be canonicalized "on the fly", one graph at a time. By standard results on V-reconstructibility (see [3]), only G need be considered, which are connected, have at most half the possible edges present, and are not regular. For each $\bar{\text{V}}$-graph, the V-graphs may be generated and canonicalized. The parts may be grouped, where in a group the size and degree sequence is the same.

Each group is assigned a distinct set of colors, and colors assigned to the nodes of a part in all possible ways. Algorithm 2.14 of [2] is useful in this step. A check is made that the decks are distinct. □

4. Computations for E-graphs

This section revises section 7 of [3]. The claims will be stated as theorems; they have already appeared in [7], More detailed proofs will be given here. Recall from [3] that a graph G is said to be E-reconstructible if every edge coloring of G is reconstructible. Recall also that the multiset of colored edges is reconstructible, whence the multiset incident to the vertex v is known for G_v. From hereon let G denote an edge coloring of K_n.

Theorem 9. *K_3 is E-reconstructible.*

Proof. G is reconstructible from any G_v by adding the other two edges. □

Theorem 10. *K_4 is E-reconstructible.*

Proof. The proof may be divided into cases.

Case T1, there is a monochromatic triangle. The remaining edges may be added arbitrarily.

Case S1, there is a monochromatic star. The remaining edges may be added arbitrarily.

Case T3, there is a 3 colored triangle. Let 123 be the colors and xyz the colors of the other 3 edges, the complementary star. The other 3 stars are colored 12x, 13y, and 23z. If these are distinct sets then G is readily reconstructed. Otherwise, w.l.g. x=3 and y=2. Whether or not z=1 G is readily reconstructed.

Case S3, there is a 3 colored star. This is similar to case T3, with stars and triangles interchanged.

In the remaining case, there is a 112 star and an xyz triangle, where in the other 3 triangles 12x, 12y, 11z, x and y are 1 or 2 and z is 2 or 3. Both the cases z=3 and z=2 are readily reconstructible. □

Theorem 11. *K_5 is E-reconstructible.*

Proof. Let P be a partition of n_e, the number of edges. Assign n_i colors to part i, where n_i is the value of part i. Let G be K_n with a partition Q of the edges, with part size list P. Let S_P be the set of canonicalized such G (writing a file of these may be done first). For $G \in S_P$ with set partition Q let T_Q be the set of edge colorings of K_n which agree with the colors assigned to P. It suffices to verify by computer that for each P, the graphs in $\cup_Q T_Q$ have distinct decks.

As a preliminary step, the number partitions 1^{10}, 21^8, 31^7, and $2^2 1^6$ may be omitted, since G may be seen to be reconstructible in these cases. Indeed, there is a vertex v such that in G_v the edge colors are distinct and there is an edge incident to v whose color is not one of these. G may be reconstructed from G_w where w is a vertex other than v. □

As noted in [3] even enumerating the set partitions of a 15 element set requires a fairly extensive computation. Further discussion of K_6 is omitted.

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