Rigorous Vector Calculus

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1. Introduction.

Vector calculus is a classic example of a subject where there is a division between theory and application. Students need to master vector calculus early, in order to use it in calculations in a variety of subjects, in particular basic physics.

On the other hand students with an interest in rigor benefit from placing the study of vector calculus on a firm theoretical foundation. This serves as an intellectual exercise, which develops new skills and interests, even in students more interested in applications.

The purpose of this text is to provide a rigorous treatment of vector calculus, for interested second year undergraduates. This distinguishes it from various other texts which cover rigorous vector calculus, which typically are for third or fourth year. Examples of such may be found in the references.

The text covers various preliminary topics in a self-contained manner, including continuity and limits, linear algebra, topology, matrices, and measure and integration. Indeed, it can serve as a brief introduction to these topics for beginning students. A treatment is also given of complex numbers and transcendental functions; properties of the latter for example are needed to compute the volume of a sphere.

The remainder of this chapter will cover some general background. Some notation from informal set theory which will be used in the text includes the following. The "Cartesian product" $S_1 \times \cdots \times S_n$ of n sets $S_1, \ldots, S_n$ is the set of "ordered n-tuples" $\langle x_1, \ldots, x_n \rangle$ where $x_i \in S_i$. The Cartesian product of n copies of S with itself (the set of ordered n-tuples of elements of S) is denoted S^n. The Cartesian product can be defined for an infinite family of sets; this will not be needed in this text.

A function $f : x \mapsto y$ is a set of ordered pairs $\{\langle u, v \rangle\} \subseteq x \times y$, where for each $u \in x$ there is exactly one $v \in y$ such that $\langle u, v \rangle \in f$. This v is denoted $f(u)$. If $x' \subseteq x$ then $f[x'] = \{v \in y : f(u) = v$ for some $u \in x'\}$. If $y' \subseteq y$ then $f^{-1}[y'] = \{u \in x : f(u) = v$ for some $v \in y'\}$. The domain $\mathrm{Dom}(f)$ is x. The range $\mathrm{Ran}(f)$ is $f[x]$.

The notation ι will be used to denote the identity function $\iota(u) = u$, on a set x which frequently is self-evident from context.

$\mathcal{R}$ will be used to denote the real numbers. These may be described as the unique ordered field having the least upper bound property (see [DowdBG]). An "open interval" in $\mathcal{R}$ is a set $\{x : a < x < b\}$ for some $a, b \in \mathcal{R}$ with $a < b$; the notation (a, b) is often used to denote the interval. A "closed interval" $\{x : a \leq x \leq b\}$, where $a \leq b$, is denoted $[a, b]$.

The following notation will be used for commonly considered subsets of $\mathcal{R}$; formal definitions may be found in [DowdBG] for example.

$\mathcal{N}$ denotes the nonnegative integers.

$\mathcal{Z}$ denotes the integers.

$\mathcal{Q}$ denotes the rational numbers.

The definitions of $\mathcal{R}$, $\mathcal{Z}$, etc., given in [DowdBG] make use of basic concepts from abstract algebra. These will occasionally be required later in the text. A brief self-contained treatment is given in the additional material.

Additional material.

A structure in mathematics is given as a set, with some constants, relations, and functions defined on it. Properties which these operations must satisfy may be give as axioms. A list of common types of structures, and their axioms, will be given. The binary relation of equality, for which the symbol "=" is used, is considered to be defined in any structure.

Any of various introductory texts can be consulted for further details concerning the types of structures listed below, and discussions of structures in general. For the latter, see [wikiUnivAlg].

A group is a structure with a binary function $\cdot$ (multiplication) and a constant 1 (the identity) satisfying the following axioms.

- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (associative law)
- $x \cdot 1 = 1 \cdot x = x$ (1 is an identity)
- For all x there exists y such that $x \cdot y = y \cdot x = 1$ (existence of an inverse)

Although details are omitted, various basic laws follow from the axioms. For example, an identity element is unique, and so is an inverse element; the inverse of x is denoted x^{-1}. The general linear group (see chapter 7) is an example of a group.

A commutative group is a group satisfying the following additional axiom.

- $x \cdot y = y \cdot x$ (commutative law).

In various cases the symbol "+" is used rather than "$\cdot$", and the operation is called addition rather than multiplication. In such cases, 0 is written for the identity element, and $-x$ for the inverse element. $\mathcal{Z}$ with addition is an example of a commutative group.

A ring is a structure with operations of multiplication and addition, such that addition satisfies the axioms of a commutative group, multiplication is associative and has an identity, and multiplication distributes over addition. In detail,

- $(x + y) + z = x + (y + z)$
- $x + 0 = 0 + x = x$
- For all x there exists y such that $x + y = y + x = 0$
- $x + y = y + x$

- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- $x \cdot 1 = 1 \cdot x = x$
- $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$ (distributive law).

The $n \times n$ matrices of real numbers (or rational numbers, or integers), with matrix addition and multiplication (see chapter 4) is an example of a ring.

A commutative ring is a ring satisfying the following additional axiom.

- $x \cdot y = y \cdot x$.

$\mathcal{Z}$ with addition and multiplication is an example of a commutative ring.

A field is a commutative ring which satisfies the following additional axioms.

- For all x, if $x \neq 0$ then there exists y such that $x \times y = y \times x = 1$
- $0 \neq 1$

$\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{C}$ (the complex numbers, defined in chapter 9) are examples of fields.

A vector space, or linear space, over a field F is a set of elements V, called vectors, with a binary function $+$ defined on V, called vector addition; a zero vector 0; and a function from $F \times V$ to V, called scalar multiplication, which satisfies the following axioms, where x, y, z are vectors and b, c are scalars (elements of F).

- V is a commutative group under addition, with identity 0.
- $1x = x$;
- $(bc)x = b(cx)$;
- $c(x + y) = cx + cy$;
- $(b + c)x = bx + cx$.

The axioms regarding scalar multiplication characterize the "action" of multiplication by a scalar on the additive group of vectors. In the next chapter it will be seen that Euclidean space, the space of vector calculus, is a real vector space (i.e., the scalar field is $\mathcal{R}$).

Suppose V is a vector space over a field F, and S is a set. Let V^S be the set of functions $f : S \mapsto V$. This may be made into a vector space over F, with the operations of "pointwise" addition ($(f_1 + f_2)(x) = f_1(x) + f_2(x)$) and scalar multiplication ($(cf)(x) = cf(x)$). For readers familiar with infinite Cartesian products, the vector space V^S is just the Cartesian product of S copies of V. An example of a vector space such as this will be given in chapter 18.

A module M over a commutative ring R is defined exactly as a vector space over a field F. The absence of a multiplicative inverse in R gives rises to differences between the theory of vector spaces and the theory of modules. For example $\{2\}$ is a maximal linearly independent subset of $\mathcal{Z}$ as a module over $\mathcal{Z}$; but it does not generate the module.

For any of the above types of structures (indeed for types of structures with no relations other than equality), a subset S' of the structure S, which contains the constants and is closed under the functions, is said to be a substructure (subgroup, subring, subspace, submodule, etc.) For example $\mathcal{Q}$ is a subfield of $\mathcal{R}$, and $\mathcal{Z}$ is a subring of $\mathcal{Q}$ considered as a ring.

Suppose R is a ring and S is a set. The set R^S of functions $f : S \mapsto R$ may be made into an R-module exactly as for vector spaces. The set of functions which are nonzero only finitely often is readily verified to be a submodule. This module is called the free R-module generated by S; it provides a rigorous definition of the "set of formal linear combinations over R of elements of S". An example of such a module will be seen in chapter 18.

Another general concept regarding algebraic structures (for simplicity having no relations other than equality) is that of a homomorphism. Given structures S, T of the same type, a homomorphism is a function $h : S \mapsto T$ "preserving" the constants and functions, that is, with obvious notation, such that $h(c_S) = c_T$ for each constant c, and $h(f_S(x_1, \ldots, x_n)) = f_T(h(x_1), \ldots, h(x_n))$ for each function f. An isomorphism is a bijection h such that h and h^{-1} are both homomorphisms. An example of a homomorphism will be given in chapter 15.

2. Euclidean space.

In the case of the Cartesian product $\mathcal{R}^n$ of n copies of the real numbers $\mathcal{R}$, the elements are called "vectors". A vector is an ordered n-tuple $\langle r_1, \ldots, r_n \rangle$ of n real numbers $r_i \in \mathcal{R}$.

The name "n-dimensional Euclidean space" may be used for $\mathcal{R}^n$. It is a principle object of study in both mathematics and physics. As such, as might be expected, it has many basic properties which are shared by "other" spaces of mathematics and physics (for example "differentiable manifolds", of which $\mathcal{R}^n$ is the archetypal example).

For one example, $\mathcal{R}^2$ may be characterized "geometrically", as the unique system of points and lines, satisfying "Hilbert's axioms", a certain system of axioms concerning incidence, betweenness, equidistance, and a "completeness" axiom. A discussion of Hilbert's axioms may be found in appendix 1 of [DowdBG].

More importantly, $\mathcal{R}^n$ is a "normed linear space", a type of "metric space", which in turn is a type of "topological space" (these facts will be proven in theorems 1 to 5 below). Facts about $\mathcal{R}^n$ may be seen in perspective as facts about such spaces.

A topological space is a set X of points, with certain subsets of X designated as the "open subsets". The following axioms must be satisfied.

1. The empty set $\emptyset$ and the entire space X are open subsets.
2. If C is a collection of open subsets then $\bigcup C$ is an open subset.
3. If C is a finite collection of open subsets then $\bigcap C$ is an open subset.

Although the notion of a topological space is quite general, it has turned out to be a fundamental one. It dates from the early 20th century.

Requirement (1) is redundant, considering $\emptyset$ to be the empty union, and X the empty intersection. Alternatively, the axioms may be stated as requiring openness of X, closure under arbitrary union, and closure under pairwise intersection.

The collection T of open subsets is called the "topology" of the space. A subset B of T is said to be a "base" for the topology if every open subset can be written as a union of open subsets which are elements of B. Open subsets which are elements of B are called "basic open subsets".

Theorem 1. A set B of subsets of a set X is a base for a topology on X iff

1. $\bigcup B = X$, and
2. for all $U, V \in B$ and all $x \in U \cap V$ there is a $W \in B$ such that $x \in W$ and $W \subseteq U \cap V$.

Proof: Let T be the subsets U such that $U = \bigcup C$ for some subcollection $C \subseteq B$. Suppose T is a topology. Condition (1) is satisfied by the definition of a topology. Suppose $U, V \in B$ and $x \in U \cap V$. Since $U \cap V \in T$ by the assumption that T is a topology, $U \cap V = \bigcup C$ for some subcollection $C \subseteq B$. It follows that $x \in W$ for some $w \in C$.

Conversely, suppose conditions (1) and (2) are satisfied. By set theory, T is closed under unions of subcollections. By condition (1), $X \in T$. Suppose $U, V \in T$. For each $x \in U \cap V$ choose W_x as in condition (2); then $U \cap V = \bigcup_x W_x$, whence $U \cap V \in T$. QED.

A metric space is a set X of points, with a binary function d defined on it, which is called the "metric" or "distance" function. The following axioms must be satisfied.

1. $d(x, y) = d(y, x)$.
2. $d(x, y) \geq 0$.
3. $d(x, y) = 0$ iff $x = y$.
4. $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality).

Let $B_{x\epsilon} = \{y : d(y, x) < \epsilon\}$; such sets are often called "open balls".

Theorem 2. In a metric space X, the open balls form the base for a topology on X.

Proof: Suppose that $x \in B_{u\zeta} \cap B_{v\xi}$. Then $d(u, x) < \zeta$ and $d(v, x) < \xi$, so there is a nonzero ϵ such that $\epsilon < \zeta - d(u, x)$ and $\epsilon < \xi - d(v, x)$. Suppose $y \in B_{x\epsilon}$ Then $d(x, y) < \epsilon < \zeta - d(u, x)$, whence $d(u, y) \leq$

$d(u,x)+d(x,y) < d(u,x)+(\zeta - d(u,x)) = \zeta$, whence $y \in B_{u\zeta}$. This argument shows that $B_{x\epsilon} \subseteq B_{u\zeta}$. Similarly $B_{x\epsilon} \subseteq B_{u\zeta}$. Since $x \in B_{x,\epsilon}$ for any $\epsilon > 0$, conditions (1) and (2) of theorem 1 are satisfied. QED.

The topology determined by a metric is often called the "metric topology".

Various "component-wise" operations may be defined on $\mathcal{R}^n$, as follows.

$\langle x_1,\ldots,x_n\rangle + \langle y_1,\ldots,y_n\rangle = \langle x_1+y_1,\ldots,x_n+y_n\rangle$.

The zero vector is $\langle 0,\ldots,0\rangle$.

$r\langle x_1,\ldots,x_n\rangle = \langle rx_1,\ldots,rx_n\rangle$.

Recall the definition of a vector space from chapter 1; when the field of scalars is $\mathcal{R}$, the vector space is said to be a real vector space.

Theorem 3. When equipped with the component-wise operations just defined, $\mathcal{R}^n$ is a real vector space.

Proof: This is a special case of the theorem that the component-wise operations define a vector space structure on the product of any family of vector spaces over a field F. A proof for $\mathcal{R}^n$ is left to exercise A1. QED.

If V is a real vector space, a norm on V is a function from V to $\mathcal{R}$ (denoted $|x|$), which satisfies the following axioms.

1. $|x| \geq 0$;
2. $|x| = 0$ iff $x = 0$;
3. $|cx| = |c||x|$;
4. $|x+y| \leq |x| + |y|$.

In axiom 3, $|c|$ is the absolute value of c (the absolute value function is defined in chapter 8 of [DowdBG]). A real vector space which is equipped with a norm is called a normed linear space.

For the next definition, the reader is assumed to be familiar with basic facts concerning the square root function. In particular, $x^2 \geq 0$; and if $r \geq 0$ there is a unique $s \geq 0$, denoted $\sqrt{r}$, such that $s^2 = r$. Additional material at the end of the chapter contains a discussion of roots, and of additional facts, which will be relied on later without comment.

To define a norm on $\mathcal{R}^n$, it is convenient first to introduce the "inner product" or "dot product", a function from $\mathcal{R}^n \times \mathcal{R}^n$ to $\mathcal{R}$. This is defined by the formula $x \cdot y = x_1y_1 + \cdots + x_ny_n$, where $x = \langle x_1,\ldots,x_n\rangle$ and $y = \langle y_1,\ldots,y_n\rangle$. Let $|x|$ denote $\sqrt{x \cdot x}$, or $\sqrt{x_1^2 + \cdots + x_n^2}$.

Theorem 4.

a. $|x \cdot y| \leq |x||y|$, (Cauchy-Schwarz inequality).
b. The function $|x|$ is a norm on $\mathcal{R}^n$.

Proof: It is easily seen that

$$0 \le \left| \frac{x}{|x|} \pm \frac{y}{|y|} \right|^2 = 2 \pm 2 \frac{x}{|x|} \cdot \frac{y}{|y|},$$

and part a follows. It is easily seen that if $r \neq 0$ then $\sqrt{r} \neq 0$; axioms 1 and 2 follow. Axiom 3 follows from the fact that for $r, s \in \mathcal{R}$, $|rs| = |r||s|$. For axiom 4, both sides are nonnegative, so squaring both sides yields an equivalent relation; after canceling terms, it remains to show that $x \cdot y \le |x||y|$. This follows by the Cauchy-Schwarz inequality. QED.

The norm of theorem 4 is called the "Euclidean norm". Note that, since $\sqrt{x^2} = |x|$, the absolute value is the Euclidean norm on $\mathcal{R}^1$, which is basically the same thing as $\mathcal{R}$.

Theorem 5. If V is a real vector space and $|x|$ is a norm on V then the function $d(x, y) = |x - y|$ is a metric on V.

Proof: Exercise 2. QED.

From here on, $\mathcal{R}^n$ denotes the normed linear space, which may considered as a metric space or a topological space according to the facts given above. The topology determined by the open balls is called the "usual topology". As already mentioned, a property of $\mathcal{R}^n$ may be a property of topological or metric spaces, which holds for the particular space $\mathcal{R}^n$. Examples will be seen in the next chapter.

Additional material.

In the axiomatic characterization of the real numbers, the laws obeyed by $+$ and $\cdot$ are specified by the axioms. Additional functions may be defined, and their properties proved, from the axioms. A principal example is the power function x^r, where x is a real number with $x > 0$, and r is any real number.

The definition of x^r proceeds in stages, first giving the definition of x^n for $n \in \mathcal{N}$. This may be defined as the unique function such that $x^0 = 1$, and $x^{n+1} = x \cdot x^n$. Basic properties of this function include the following.

Lemma A1.

a. $x^{m+n} = x^m x^n$ (addition law).
b. $x^{mn} = (x^m)^n$ (multiplication law).
c. $(xy)^n = x^n y^n$.
d. If $x > 1$ and $n > m$ then $x^n > x^m$.
e. $1^n = 1$.
f. If $0 < x < 1$ and $n > m$ then $x^n < x^m$.
g. If $x > y > 0$ and $n > 0$ then $x^n > y^n$.

Proof: Parts a to e are proved by induction on n; details are left to the reader. That $(1/x)^n = 1/x^n$ follows, since $x^n(1/x)^n = (x \cdot 1/x)^n =$

$1^n = 1$. Part f then follows, since $1/x > 1$, so $1/x^n > 1/x^m$, so $x^n < x^m$. Part g follows by induction, as does $x^n > 0$ if $x > 0, n > 0$ (exercise). For part h, note that $x^n/y^n = (x/y)^n > 1^n = 1$ follows from $x/y > 1$, and $y^n > 0$ was just shown. QED.

There is a unique extension of the function x^n for $x \in \mathcal{N}$ to a function x^n for $n \in \mathcal{Z}$, which satisfies the addition law. Namely, for $n < 0$, let $x^{-n} = 1/x^n$. Lemma A1 holds for this function, and also

h. If $x > y > 0$ and $n < 0$ then $x^n < y^n$.

This follows by arithmetic; details are left to exercise 1.

Theorem A2. If x is a nonnegative real number and n a positive integer then there is a unique nonnegative real number y such that $y^n = x$.

Proof: To prove the existence of y, first note that $0^n \leq x$, and if $w > x$ and $w \geq 1$ then $w^n > x$, so $S = \{w \geq 0 : w^n \leq x\}$ is nonempty and bounded above. It therefore has a sup y. We claim that $y^n = x$. Firstly, u is an upper bound to S if and only if $u^n \geq x$; if $w^n < x$ then w is not an upper bound (because there is a $w' > w$ with $w'^n < x$), and if $u^n \geq x$ then u is an upper bound, since then $w > u$ implies $w^n > u^n$, so $w^n > x$. Secondly, if $u^n > x$ then u is not the least upper bound (because there is a $u' < u$ with $u'^n > x$; details are left to exercise A2).

Uniqueness of y is immediate from the observation that $0 \leq w < v$ implies $w^n < v^n$, and so if w and v are distinct nonnegative real numbers their n-th powers are distinct. QED.

The value y of the theorem is called the n-th root of x and denoted $\sqrt[n]{x}$, or $x^{1/n}$. If $n = 2$ y is called the square root and denoted $\sqrt{x}$; if $n = 3$ y is usually called the cube root (rather than the third root).

There is a unique extension of the function x^n for $x \in \mathcal{Z}$ to a function x^q for $q \in \mathcal{Q}$, which satisfies the multiplication law. Namely, for $n > 0$, let $x^{m/n} = (x^m)^{1/n}$. In fact, the extension is uniquely determined by the requirements $(x^{1/n})^n = 1$ and the addition law. Lemma A1 holds for this function, including the additional clause h. This follows by arithmetic; details are left to exercise A3.

The power function may be extended uniquely to real exponents, "by continuity". Any real r equals $\sup\{q : q < r\}$; the value x^r for $x > 1$ may be given as $\sup\{x^q : q < r\}$. Lemma A1 continues to hold. Further discussion is left to the additional material of chapter 6.

Definitions are commonly given in some cases for $x < 0$. For a real number $r \geq 0$, 0^r can be defined to be 0 if $r > 0$, and 1 if $r = 0$. However, 0^r cannot be defined for $r < 0$, so that the addition law holds. For $n \in \mathcal{N}$, $(-1)^n$ can be defined to be $+1$ for even n, and -1 for odd n; $(-x)^n$ for $x > 0$ can be defined to be $(-1)^n x^n$.

Exercises.

1. Prove theorem 3. Hint: verify the axioms using the definition and the fact that $\mathcal{R}$ is a field.

2. Prove theorem 5. Hint: Use the axioms for a vector space and for a norm to derive the axioms for a metric.

A1. Prove the stated properties of the function x^n for $n \in \mathcal{Z}$.

A2. Suppose n is an integer and $x, \delta, w, \epsilon \in \mathcal{R}$. Show the following.
a. If $x \geq 2$ then $x + 2 \leq 2x$.
b. If $n \geq 1$ then $2^n \geq 2$. Hint: use mathematical induction.
c. If $n \geq 1$ then $2^n + 2 \leq 2^{n+1}$.
d. If $n \geq 1$, $\delta \geq 0$, and $\delta < 1$ then $(1+\delta)^n < 1 + 2^n\delta$. Hint: use c and induction.
e. If $w \geq 0$ and $w^n < x$ then there is an $\epsilon > 0$ such that $(w+\epsilon)^n < x$. Hint: If $w = 0$ choose $\epsilon < 1, \epsilon < x$. Otherwise divide both sides by w^n, and apply d with $\delta = \epsilon/w$ where $\delta < 1$ and $\delta < (x/w^n - 1)/2^n$.
f. If $w \geq 0$ and $w^n > x$ then there is an $\epsilon > 0$ such that $(w-\epsilon)^n > x$. Hint: First show $(1-\delta)^n > 1 - 2^n\delta$ for $n \geq 1$, $0 \leq \delta < 1$.

A3. Prove the stated properties of the function x^q for $q \in \mathcal{Q}$.

3. Continuity and limits.

The notion of continuity may be considered in topological spaces. The notion of a limit may be considered in metric spaces. Taking this approach has various advantages over considering the notions just for $\mathcal{R}^n$, and is not much more involved.

A topological space is often indicated by giving its underlying set X, omitting a name for the topology. If a name for the topology is of use, it may be given explicitly.

If X is a topological space, with topology T, and $X' \subseteq X$, the "subspace topology" on X' is the set $\{U \cap X' : U \in T\}$. This is readily verified by set theory to be a topology on X'. X', equipped with the subspace topology, is said to be a subspace. It is easy to check that if B is a base for a topology T in a space X, and $X' \subseteq X$, then $\{U \cap X' : U \in B\}$ is a base for the subspace topology.

For an example of a subspace, it is easily seen that an open ball in $\mathcal{R}$ is exactly the same thing as an open interval. Thus, by theorem 1.2, the open intervals are a base for the usual topology on $\mathcal{R}$. It is easy to check that the intervals (a', b') with $a \leq a' < b' \leq b$ form a base for the subspace topology on the subspace (a, b) of $\mathcal{R}$.

Suppose X is a topological space, with topology T_X, and Y is a topological space, with topology T_Y. A function $f : X \mapsto Y$ is said to be continuous if $f^{-1}[V] \in T_X$ whenever $V \in T_Y$. Less formally, f is continuous if $f^{-1}[V]$ is open whenever V is; a name for the topology may be omitted, since which topology is intended may be inferred.

Theorem 1.

a. The identity function on a topological space is continuous.
b. The composition of continuous functions is continuous.
c. The restriction of a continuous function to a subspace is continuous.
d. The corestriction of a continuous function to a subspace containing the range is continuous.

Proof: Part a is trivial from the definitions. Part b follows because $(g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]]$. Part c follows because $(f \restriction X')^{-1}[V] = f^{-1}[V] \cap X'$. Part d follows because if f' is the corestriction, then $(f')^{-1}[V \cap Y'] = f^{-1}[V]$. QED.

The proof of the preceding theorem is an example of "practical advantages" of the abstract approach.

It is useful to give a notion of continuity "at a point". If $f : X \mapsto Y$ is a function between topological spaces, and $x \in X$, say that f is continuous at x iff for every open $V \subseteq Y$ containing $f(x)$ there is an open $U \subseteq X$ containing x, such that $f[U] \subseteq V$. The following convenient terminology is widely used: Suppose a base B is given for a topology T; then an element of B is called a basic open set.

Lemma 2. A function $f : X \mapsto Y$ between topological spaces is continuous iff it is continuous at x for all $x \in X$.

Proof: Suppose f is continuous. Suppose $V \subseteq Y$ is open and $f(x) \in V$. Letting $U = f^{-1}[V]$, U is open, $x \in U$, and $f[f^{-1}[V]] \subseteq V$ by set theory. Suppose f is continuous at any x. Suppose $V \subseteq Y$ is open, and $x \in f^{-1}[V]$. Choose an open subset $U \subseteq X$ with $x \in U$ and $f[U] \subseteq V$. Then $U \subseteq f^{-1}[V]$. Since x is arbitrary, it follows that $f^{-1}[V]$ is open. QED.

Lemma 3. f is continuous at x iff $(*)$ for every basic open V containing $f(x)$ there is a basic open U containing x with $f[U] \subseteq V$.

Proof: Suppose f is continuous at x. If V is basic open then it is open, so there is an open U with $x \in U$ and $f[U] \subseteq V$, so there is a basic open such U. Suppose $(*)$ holds. Suppose V is open and $f(x) \in V$. Then there is a basic open V' with $f(x) \in V'$, so there is a basic open U containing x with $f[U] \subseteq V$. QED.

Theorem 4. Suppose $f : X \mapsto Y$ is a function between metric spaces, and $x \in X$. Then f is continuous at x iff for all $\epsilon > 0$ there is a $\delta > 0$ such that $f[B_{x\delta}] \subseteq B_{f(x),\epsilon}$.

Proof: This follows by lemmas 2 and 3, since the $B_{x\epsilon}$ comprise bases for the topologies. QED.

The requirement on δ may be stated as, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$. The concept of continuity can further be characterized using the notion of a limit. This notion has various uses, and the definition is a bit technical.

Suppose X is a topological space, $X' \subseteq X$ is a subspace, and $x \in X$. A point $x \in X$ is said to be a limit point of X' iff, for every open $U \subseteq X$ with $x \in U$, $X' \cap (U - \{x\}) \neq \emptyset$. X' may equal X, in which case x is simply called a limit point. If x is not a limit point, that is, if there is an open set containing only x (equivalently if $\{x\}$ is an open subset), x is said to be an isolated point.

While the notion of a limit may be defined more generally, it will be defined here for functions between metric spaces. Note that, if X is a metric space with metric d, and $X' \subseteq X$ is a subspace, then d restricted to X' is a metric on X'. It is readily verified that the topology induced on X' by the restriction of d is the subspace topology of the topology determined by d on X.

Suppose X, Y are metric spaces, $X' \subseteq X$ is a subspace, $f : X' \mapsto Y$, and $x \in X$ is a limit point of X'. The point $y \in Y$ is said to be the limit of $f(x')$ as x' approaches x iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $f[B_{x\delta} - \{x\}] \subseteq B_{y\epsilon}$. The fact that this is the case is commonly written as $\lim_{x' \to x} f(x') = y$. Note that X' is the domain of f; this may or may not be all of X. The open ball $B_{x\delta}$ is that of the subspace topology, i.e., $\{x' \in X' : d_X(x, x') < \delta\}$, where d_X is the metric on X. The requirement on δ may be stated as, if $x' \in X'$, $d(x, x') < \delta$, and $x' \neq x$, then $d(f(x), f(x')) < \epsilon$. The extra condition $x' \neq x$ ensures that the value of f at x is irrelevant, even if $x \in X'$.

Theorem 5. There can be at most one y such that $\lim_{x' \to x} f(x') = y$.

Proof: Suppose y_1 and y_2 are two such. Let $\epsilon = d(y_1, y_2)/2$. If this is non-zero, then $f[B_{x\delta_1} - \{x\}] \subseteq B_{y_1\epsilon}$ for some δ_1. But then there can be no $\delta_2 \leq \delta_1$ such that $f[B_{x\delta_2} - \{x\}] \subseteq B_{y_2\epsilon}$. This contradiction shows that $d(y_1, y_2) = 0$, so $y_1 = y_2$. QED.

Theorem 6. Suppose $f : X \mapsto Y$ is a function between metric spaces, and $x \in X$. If x is isolated then f is continuous at x. If x is a limit point then f is continuous at x iff $\lim_{x' \to x} f(x') = f(x)$.

Proof: For the first claim, there is a δ such that $f[B_{x\delta}] = \{f(x)\}$. For the second claim, since $y = f(x)$ the restriction $x' \neq x$ is unnecessary in the definition of the limit. QED.

The first claim of theorem 6 is a technical detail, which is often irrelevant since in common cases there are no isolated points. $\mathcal{R}^n$ for example has no isolated points; this is shown in exercise 1, using the inequality of the following theorem. Let $|x|_{\max}$ denote $\max\{|x_1|, \ldots, |x_n|\}$.

Theorem 7. $|x|_{\max} \leq |x| \leq \sqrt{n} \cdot |x|_{\max}$.

Proof: Squaring both sides yields an equivalent inequality. Now, $|x_i|^2 = x_i^2 \leq \sum_i x_i^2$. Since this holds for all i, $(\max_i |x_i|)^2 = \max_i |x_i|^2 \leq$

$\sum_i x_i^2$. In addition, $\sum_i x_i^2 \leq n \cdot (\max_i |x_i|)^2$. QED.

There is another useful application of this inequality. Two metrics d_1 and d_2 on a space X are said to be

- equivalent if they determine the same topology, and
- strongly equivalent if there are constants $a, b > 0$ such that for all x, y, $d_1(x, y) \leq a d_2(x, y)$ and $d_2(x, y) \leq b d_1(x, y)$.

Two norms $|x|_1$ and $|x|_2$ on a linear space X are said to be

- equivalent if there are constants $a, b > 0$ such that for all x, $|x|_1 \leq a|x|_2$ and $|x|_2 \leq b|x|_1$.

Let $|x|_{\text{sum}}$ denote $|x_1| + \cdots + |x_n|$.

Theorem 8.

a. The three relations above are equivalence relations.
b. Strongly equivalent metrics are equivalent.
c. Equivalent norms determine strongly equivalent metrics.
d. Two norms are equivalent iff they determine the same topology.
e. Two norms are equivalent iff there is a constant $a > 0$ such that for all x, $|x|_1 \leq a|x|_2$,
f. $|x|_{\max}$ is a norm on $\mathcal{R}^n$, equivalent to $|x|$.
g. $|x|_{\text{sum}}$ is a norm on $\mathcal{R}^n$, equivalent to $|x|$.

Proof: Exercise 2. QED.

The following “pointwise” version of theorem 1 will be needed in chapter 5.

Theorem 9.

a. If $f : X \mapsto Y$ is continuous at x and $g : Y \mapsto Z$ is continuous at $f(x)$ then $g \circ f$ is continuous at x.
b. If $f : X \mapsto Y$ is continuous at x, $X' \subseteq X$ is a subspace, and $x \in X'$ then $f \restriction X'$ is continuous at x.

Proof: Exercise 3. QED.

The notions of an infinite sequence in a metric space X, and the limit of such, will also be needed later in the text. An infinite sequence is simply a function $f : \mathcal{N} \mapsto X$; such may be written as $\langle x_n : n \in \mathcal{N} \rangle$, or simply $\langle x_n \rangle$.

The point $x \in X$ is said to be the limit of the sequence $\langle x_n \rangle$ as n approaches infinity, written $\lim_{n \to \infty} x_n = x$, iff for every $\epsilon > 0$ there exists $n \in \mathcal{N}$ such that if $n > N$ then $|x_n - x| < \epsilon$.

A suitably modified version of theorem 5 may be proved by suitably modifying the proof. Alternatively, $\mathcal{N} \cup \{\infty\}$ can be equipped with a metric, yielding the definition as a particular case of the general definition, by letting $d(x, y)$ equal $|v(x) - v(y)|$ where $v(N) = 1/(N + 1)$ and $v(\infty) = 0$. The verification is left as exercise 4.

Exercises.

1. Show the following.
a. For $x \in \mathcal{R}$ and $\epsilon > 0$ there is a rational number $q \neq x$ such that $|x - q| < \epsilon$. Hint: By the assumed characterization of $\mathcal{R}$, there is a q with $x - \epsilon < q < x$.
b. For $x \in \mathcal{R}^n$ and $\epsilon > 0$ there is is a vector $q = \langle q_1, \ldots, q_n \rangle$ of rational numbers such that $q \neq x$ and $|x - q| < \epsilon$. Hint: Use part a with bound ϵ/n and theorem 7.

2. Prove theorem 8. Hint: For part f, first verify that $|x|_{\text{max}}$ is a norm. Use theorem 7 for equivalence. For equivalence in part g, show that $|x|_{\text{max}} \leq |x|_{\text{sum}} \leq n \cdot |x|_{\text{max}}$.

3. Prove theorem 9.

4. Prove that the function $d(x, y)$ on $\mathcal{N} \cup \{\infty\}$ defined above is a metric, and that the two definitions of the limit agree.

4. Linear algebra.

As usual, the linear algebra of $\mathcal{R}^n$ may be seen as a particular case of the linear algebra of a vector space X over a field F. Recall that a subspace Y of X is a subset which, with the operations $0, +, \cdot$ inherited from X, is itself a vector space over F. It is readily verified that it suffices that Y be nonempty and closed under $+$ and $\cdot$ (exercise 1).

A linear combination of elements from a nonempty subset $S \subseteq X$ is a value of the form $c_1 x_1 + \cdots + c_n x_n$ where $n > 0$, $x_i \in S$ and $c_i \in F$ for $1 \leq i \leq n$, and the x_i are distinct.

Theorem 1. If $S \subseteq X$ is nonempty then the set of linear combinations of elements of X is the smallest subspace of X containing S.

Proof: Let Y denote the set of linear combinations. Clearly, if Z is a subspace with $S \subseteq Z$ then $Y \subseteq Z$. On the other hand, Y is closed under $+$ and $\cdot$. QED.

In the circumstances of the theorem, Y is called the subspace generated by S. A linear combination $c_1 x_1 + \cdots + c_n x_n$ is called trivial if all the c_i are 0, otherwise nontrivial. A nonempty set S of vectors is called linearly independent if no nontrivial linear combination of its elements equals 0. Note that if S is linearly independent then $0 \notin S$.

Lemma 2. Suppose $S \subseteq X$ is a linearly independent subset of size $n > 0$, T is a subset of size $n + 1$, and $S \cup \{t\}$ is linearly dependent for each $t \in T$. Then T is linearly dependent.

Proof: By induction on n. Let $S = \{s_1, \ldots, s_n\}$ and let $T = \{t_1, \ldots, t_{n+1}\}$. If $n = 1$ then $a_1 t_1 + b_1 s_1 = 0$ and $a_2 t_2 + b_2 s_1 = 0$ for some $a_i, b_i \in \mathcal{R}$; then $a_1 b_2 t_1 - a_2 b_1 t_2 = 0$. For $n > 1$, by hypothesis, for each i with $1 \leq i \leq n + 1$, there are $a_i, b_{ij} \in R$ such that $a_i t_i + \sum b_{ij} s_j = 0$. Certainly $a_{n+1} \neq 0$, so there is a j so that $b_{n+1,j}$ is nonzero. For $1 \leq i \leq n + 1$ let $E_i = a_i t_i + \sum_j b_{ij} s_j$ be a nontrivial linear combination involving t_i and equaling 0. For $1 \leq i \leq n$ let $E'_i = b_{n+1,j}E_i - b_{ij}E_{n+1}$. Then E'_i is a linear combination involving $t'_i = b_{n+1,j}a_it_i - b_{ij}a_{n+1}t_{n+1}$ and the s_k for $k \neq j$; and equaling 0. By induction the t'_i are linearly dependent, whence the t_i are. QED.

If $S \subseteq X$ is linearly independent, and the subspace generated by S equals X, S is said to be a basis for X.

Theorem 3. Suppose S is a maximal linearly independent subset of the vector space X, and $|S| = n$. Suppose T is a linearly independent set.

a. $|T| \leq n$.
b. T is maximal iff $|T| = n$.
c. T is a basis iff it is maximal.
d. For every element $y \in X$ there is a unique linear combination of elements of S such that $y = c_1x_1 + \cdots + c_nx_n$.

Proof: For part a, suppose $|T| > n$. Choose a subset $T' \subseteq T$ with $|T'| = n + 1$. By the hypothesis that S is maximal, the hypothesis of lemma 3 holds, whence T' is linearly dependent, whence T is. For part b, if T were maximal and had size $m < n$ then $|S| \leq m$ would have to hold by part a, a contradiction. For part c, if T is maximal and $x \notin T$ then there is a nonzero $a \in F$ and a $y \in Y$ such that $ax+y = 0$, where Y is the subspace generated by T; but then $x = -t/a$ is in Y. Conversely if T is a basis then it is linearly independent, and cannot be enlarged since any other element is a linear combination. For part d, since S is a basis there is some linear combination, and if there were two distinct linear combinations, then their difference would be a nontrivial linear combination with value 0. QED.

Under the circumstances of the theorem, a vector space is said to be finite dimensional, of dimension n. The notion of infinite dimension can be considered, but is not needed for vector calculus. Just as in theorem 1.3, F^n may be enriched from a set to a vector space by equipping it with the component-wise operations. The "standard unit vectors" e_i for $1 \leq i \leq n$ comprise a basis (exercise 2), where e_i is the vector which is 1 in the i-th component and 0 in the other components.

For example, $\mathcal{R}^n$ is an n-dimensional vector space. In $\mathcal{R}^3$, a single vector generates a 1-dimensional subspace, which may be seen to be a straight line though the origin. Adding a second vector not on the line yields a linearly independent set which generates a 2-dimensional subspace, which can be seen to be a copy of the "Euclidean plane" (see appendix 1 of [DowdBG]). Adding a third vector not on the plane yields a linearly independent set which generates the entire space.

For many types of structures, the Cartesian product may be enriched to form a structure of the given type. This may be called the

Cartesian product of the structures. The Cartesian product $V \times W$ of two vector spaces may be made into a vector space, with componentwise operations. Note that F^n is the Cartesian product of F (as a vector space over itself) with itself, n times. $F^n \times F^m$ is isomorphic by an obvious isomorphism to F^{n+m}.

A function $f : X \mapsto Y$ where X and Y are vector spaces over the field F said to be a linear transformation if it satisfies the following conditions:

a. $f(0) = 0$,
b. $f(x + y) = f(x) + f(y)$,
c. $f(cx) = cf(x)$.

It is easily seen that condition a is redundant.

Suppose $\{u_1, \ldots, u_n\}$ is a basis for X and $\{v_1, \ldots, v_m\}$ is a basis for Y. Then for $1 \leq j \leq n$, $f(u_j) = \sum_i f_{ij} v_i$ for some elements $f_{ij} \in F$ for $1 \leq i \leq m$. The ensemble f_{ij} of field elements goes under the name of a "matrix". Formally, this may be taken to be a function from $\{1, \ldots, m\} \times \{1, \ldots, n\}$ to F. Informally, it may be considered to be a "rectangular array" of field elements, with the entry f_{ij} being in the ith row and jth column. The notation $[f_{ij}]$ will be used to denote the matrix. The map taking a linear transformation to its matrix may be seen to be bijective (exercise 3).

Lemma 4. With notation as above, suppose $y = f(x)$ where $x = \sum_j x_j u_j$ and $y = \sum_i y_i v_i$. Then $y_i = \sum_j f_{ij} x_j$.

Proof: $y = f(x) = f(\sum_j x_j u_j) = \sum_j x_j f(u_j) = \sum_j x_j \sum_i f_{ij} v_j = \sum_i \sum_j f_{ij} x_j v_i$. By the uniqueness of the components of y,
$y_i = \sum_j f_{ij} x_j$. QED.

Theorem 5. Suppose X, Y, Z are finite-dimensional vector spaces. Suppose $f : X \mapsto Y$ and $g : Y \mapsto Z$ are linear transformations. Then $(g \circ f)_{ij} = \sum_k g_{ik} f_{kj}$.

Proof: Suppose $y = f(x)$ and $z = g(y) = (g \circ f)(x)$; write h for $g \circ f$, so that $z_i = \sum_j h_{ij} x_j$ where $[h_{ij}]$ is the matrix for h. Then $z_i = \sum_k g_{ik} y_k = \sum_k g_{ik} \sum_j f_{kj} x_j = \sum_j \sum_k g_{ik} f_{kj} x_j$. QED.

The operation on matrices of the theorem is known as "matrix multiplication". Given matrices $[g_{ik}]$ and $[f_{kj}]$, their matrix product is the matrix $[h_{ij}]$ where $h_{ij} = \sum_k g_{ik} f_{kj}$.

The vector space F^n of length n vectors has as a basis the standard unit vectors. In this case, x is just the vector $\langle x_1, \ldots, x_n \rangle$. This may be considered to be a single column matrix $[x_j]$, called a"column vector". The expression of lemma 4 then becomes a special case of matrix multiplication.

Two further definitions concerning matrices are as follows. Let I denote the identity matrix, where $I_{ij} = 1$ if $i = j$, else 0; I is readily verified to be the matrix for the identity linear transformation on F^n. The transpose of a matrix M is the matrix where $M^T_{ij} = M_{ji}$.

Recall that by set theory, a function f is surjective iff it has a right inverse. and injective iff it has a left inverse. If f has both a left inverse g_1 and a right inverse g_2 then $g_1 = g_2$ and their common value g is unique. In the case of a function between vector spaces, the following holds; it is stated only for finite dimensional spaces, but the existence of a linear inverse holds in general.

Theorem 6. Suppose $f : X \mapsto Y$ is a linear transformation from an n-dimensional space X to an m-dimensional space Y.

a. If f is injective then $n \leq m$ and there is a left inverse g which is a linear transformation.
b. If f is surjective then $m \leq n$ and there is a right inverse which is a linear transformation.
c. If $m = n$ then f is injective iff f is surjective iff f is bijective.

Proof: For part a, let B be a basis for X. It is readily verified that $f[B]$ is linearly independent; in particular $n \leq m$. Enlarge $f[B]$ to a basis C for Y; let $g(f(u)) = u$ for $u \in B$, and $g(v) = 0$ for $g \in C - f[B]$. This determines a linear transformation. For part b, let B be a basis for Y. For each $v_i \in B$ let $u_i \in X$ be such that $f(u_i) = v_i$. It is readily verified that $\{u_i\}$ is linearly independent; in particular $m \leq n$. Let $g(v_i) = u_i$; this determines a linear transformation. For part c, if f is injective then in the argument for part a, $C = f[B]$ and f is surjective. If f is surjective then in the argument for part b, $\{u_i\}$ is a basis and f is injective. QED.

In the proof of part a, let the basis B be the standard unit vectors. $f[B]$ is the (set of) columns of the matrix $[f_{ij}]$. The subspace of F^n generated by these is called the column space. Let r be its dimension; then $r \leq n$ (since $|B| = n$), $r \leq m$ (since the column space is a subspace of F^m), and f is injective iff the columns of $[f_{ij}]$ are linearly independent iff $r = n$.

The set of linear combinations of the rows of an $m \times n$ matrix M is a subspace of F^m, called the row space of the matrix. Clearly, the row space of M equals the column space of M^T. Using exercise 4, M is surjective iff M has a right inverse matrix iff M^T has a left inverse matrix iff M^T is injective. Letting s denote the dimension of the row space, it follows that $s \leq m$, $s \leq n$, and f is surjective iff the rows of $[f_{ij}]$ are linearly independent iff $s = m$.

It is left to exercise 5 to show that the dimensions of the column space and row space of a matrix are equal. Their common value is called the rank of the matrix. The matrix is said to be full rank if this is as large as possible, i.e., $\min\{m, n\}$.

For a finite set S let $|S|$ denote its size. There is no confusion with the notation $|x|$ for the norm; in the former case S is a set and in the latter x is a vector. If S is a set with $|S| = n$, a bijection $\sigma : S \mapsto S$ from S to itself is called a permutation of S. The permutations of a set S will be denoted as Sym_S, or Sym_n if $S = \{1, \ldots, n\}$. In exercise 6 it is shown that $|\mathrm{Sym}_n| = n!$, where $n!$ is defined by the recursion $0! = 1$, $(n+1)! = (n+1)n!$, and is called the factorial function.

The operations $\circ$ of composition, and $\sigma \mapsto \sigma^{-1}$, are defined on Sym_S Recalling that ι denotes the identity function, These obey the following laws.

- $(\sigma \circ \tau) \circ \rho = \sigma \circ (\tau \circ \rho)$.
- $\sigma \circ \iota = \iota \circ \sigma = \sigma$.
- $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = \iota$.

These are just the group laws, and Sym_S is a group.

Define the map $\mathrm{sg} : \mathrm{Sym}_n \mapsto \pm 1$ as follows:

$$\mathrm{sg}(\sigma) = \prod_{1 \le i < j \le n} \mathrm{sg}_\sigma(i, j)$$

where $\mathrm{sg}_\sigma(i,j) = +1$ if $\sigma(i) < \sigma(j)$ (σ preserves the order of i and j), and $\mathrm{sg}_\sigma(i,j) = -1$ if $\sigma(i) > \sigma(j)$ (σ reverses the order of i and j). The value $\mathrm{sg}(\sigma)$ is known as the "sign" of the permutation σ.

Lemma 7. $\mathrm{sg}(\sigma\tau) = \mathrm{sg}(\sigma)\mathrm{sg}(\tau)$.

Proof: Let $S_\alpha = \{(i,j) : \mathrm{sg}_\tau(i,j) = \alpha\}$ and let $S_{\alpha\beta} = \{(i,j) : \mathrm{sg}_\tau(i,j) = \alpha \text{ and } \mathrm{sg}_\sigma(\tau(i), \tau(j)) = \beta\}$, where α or β is $+$ (for $+1$) or $-$ (for -1). Then the S_α are disjoint and their union is $\{1, \ldots, n\}$; this is also true of the $S_{\alpha\beta}$. Further $\mathrm{sg}(\tau) = (-1)^{|S_-|}$, $\mathrm{sg}(\sigma) = (-1)^{|S_{+-} \cup S_{--}|}$, and $\mathrm{sg}(\sigma\tau) = (-1)^{|S_{+-} \cup S_{-+}|}$. The lemma is equivalent to

$$(-1)^{|S_- + |S_{+-} \cup S_{--}|} = (-1)^{|S_{+-} \cup S_{-+}|},$$

which follows since the exponent of the left side is

$$|S_{+-}| + 2 \cdot |S_{--}| + |S_{-+}|.$$

QED.

In exercise 7 it is shown that any permutation may be written as a product of "transpositions", where a transposition is a permutation exchanging two integers i and j. By the lemma, in any such product, the number of transpositions is even if $\mathrm{sg}(\sigma) = +1$, and odd if $\mathrm{sg}(\sigma) = -1$.

Suppose M is an $n \times n$ matrix and $\sigma \in \mathrm{Sym}_n$. Let M_σ denote

$$\prod_{i=1}^{n} M_{i,\sigma(i)}.$$

The determinant $\det(M)$ of an $n \times n$ matrix M is defined to be

$$\sum_{\sigma \in \mathrm{Sym}_n} \mathrm{sg}(\sigma) M_\sigma,$$

This definition holds for matrices with entries from any field, indeed any commutative ring, although as usual the case of interest in vector analysis is the field $\mathcal{R}$. For example if $n = 2$ then $\det(M) = M_{11}M_{22} - M_{12}M_{21}$, and if $n = 3$,

$$\begin{aligned}\det(M) = \quad & M_{11}M_{22}M_{33} - M_{11}M_{23}M_{32} + M_{12}M_{21}M_{33} \\ & -M_{12}M_{23}M_{31} + M_{13}M_{21}M_{32} - M_{13}M_{22}M_{31}.\end{aligned}$$

Lemma 8.

$$\prod_{i=1}^{n}\sum_{k=1}^{n} x_{ik} = \sum_{\pi \in \mathrm{Sym}_n} \prod_{i=1}^{n} x_{i,\pi(i)}.$$

Proof: This is an immediate consequence of the distributive law for commutative rings. To form the product of the sums, arrange the terms in a square, with the rows being the terms of a sum. Then sum the products along each "transversal" of the square, where a transversal is a selection of 1 element from each row and column; such are in bijective correspondence with the permutations. QED.

Theorem 9. $\det(MN) = \det(M)\det(N)$.

Proof: Let $L = MN$; then

$$\begin{aligned}\det(L) &= \sum_\sigma \mathrm{sg}(\sigma) L_\sigma = \sum_\sigma \mathrm{sg}(\sigma) \prod_{i=1}^{n} L_{i,\sigma(i)} \\ &= \sum_\sigma \mathrm{sg}(\sigma) \prod_{i=1}^{n}\sum_{k=1}^{n} M_{ik}N_{k,\sigma(i)} \\ &= \sum_\sigma \mathrm{sg}(\sigma) \sum_\pi \prod_{i=1}^{n} M_{i,\pi(i)} N_{\pi(i)\sigma(i)} \\ &= \sum_\sigma \sum_\pi \mathrm{sg}(\sigma) \prod_{i=1}^{n} M_{i,\pi(i)} \prod_{i=1}^{n} N_{i,\sigma\pi^{-1}(i)} \\ &= \sum_\nu \sum_\pi \mathrm{sg}(\nu\pi) \prod_{i=1}^{n} M_{i,\pi(i)} \prod_{i=1}^{n} N_{i,\nu(i)} \\ &= \sum_\nu \sum_\pi \mathrm{sg}(\nu)\mathrm{sg}(\pi) M_\pi N_\nu \\ &= \Big(\sum_\pi \mathrm{sg}(\pi) M_\pi\Big)\Big(\sum_\nu \mathrm{sg}(\nu) N_\nu\Big) = \det(M)\det(N).\end{aligned}$$

QED.

Theorem 10. M is full rank iff $\det(M) \neq 0$.

Proof: If M is full rank then M^{-1} exists, and $1 = \det(I) = \det(MM^{-1}) = \det(M)\det(M^{-1})$, so $\det(M) \neq 0$. Suppose M has rank r where $r < n$. As in exercise 5, write $M = BC$ where B is an $n \times r$ matrix and C is an $r \times n$ matrix. Let B' be B, extended with $n - r$ columns of 0's. Let C' be B, extended with $n - r$ rows of 0's. Then $M = B'C'$. But it is easily seen from the definition that if a matrix has a row or column of 0's then its determinant is 0. QED.

Exercises.

1. Show that if Y is nonempty and closed under $+$ and $\cdot$ then it is a subspace. Hint: $0 = x + (-1) \cdot x$, showing the existence of 0 and additive inverses. The remaining axioms hold because they hold in X.

2. Show that the standard unit vectors comprise a basis for $\mathcal{F}^n$.

3. Show that the map from a linear transformation f to its matrix $[f_{ij}]$ is bijective. Hint: There is an inverse map from a rectangular array $[m_{ij}]$ to a linear transformation.

4. Show that $(MN)^T = N^T M^T$,

5. Show that the dimension r of the column space and the dimension s of the row space of an $m \times n$ matrix M are equal. Hint: Let B be an $m \times r$ matrix whose columns are a basis for the column space. Then $M = BC$ where C is an $r \times n$ matrix. The row space of M is a subspace of the row space of C, and so $s \leq r$. That $r \leq s$ follows by considering M^T.

6. Show that $|\mathrm{Sym}_n| = n!$. Hint: Use induction on n.

7. Let $(i_1, \ldots, i_t)$ denote the permutation σ where $\sigma(i_j) = i_{j+1}$ if $j < t$, and $\sigma(i_t) = i_1$. Such a permutation is called a cycle. Show that a permutation can be written in an essentially unique way as a product of disjoint cycles. Show that a nontrivial cycle can be written as a product of transpositions (cycles of length 2).

5. Differentiation.

Suppose U is an open subset of $\mathcal{R}^n$, $f : U \mapsto \mathcal{R}^m$, and $x \in U$. Suppose $\phi : \mathcal{R}^n \mapsto \mathcal{R}^m$ is a linear transformation having the property that

$$\lim_{w \to x} \frac{|f(x) - f(x) - \phi(w - x)|}{|w - x|} = 0.$$

Then ϕ is said to be the derivative of f at x, and f is said to be differentiable at x.

The definition of the derivative can be made in a more general context, namely that of functions between normed linear spaces over $\mathcal{R}$

which are "complete" (completeness will be discussed in the next chapter). This is a fact of interest in important branches of mathematics, for example "the calculus of variations". For simplicity, only finite dimensional spaces will be considered here, although various facts hold more generally.

The importance of the derivative resides in the fact that it is a "linear approximation" to f at the point x. This is a great deal of valuable information. Indeed, the definition may be re-written as $f(x+\xi) \approx f(x) + \phi(\xi)$, where the degree of approximation required is made precise by the requirement that the "error term" $E(\xi) = f(x+\xi) - f(x) - \phi(\xi)$ approach 0 "faster than" ξ.

Some further facts about linear transformations which will be useful below are summarized in the next few lemmas. Some of these will be stated in greater generality than finite dimensional spaces over $\mathcal{R}$, for the usual reasons that the general statement is no harder to prove, and provides perspective. Recall that for sets X and Y, Y^X denotes $\{f : f{:}\, X \mapsto Y\}$. If Y is a linear space over a field F, define the 0 map by $f(x) = 0$, the sum $f+g$ of two maps by $(f+g)(x) = f(x)+g(x)$, and the scalar multiple cf by $(cf)(x) = c{\cdot}f(x)$. If X, Y are both linear spaces over a field F, let $L(X;Y)$ denote the set of linear transformations from X to Y.

Lemma 1. Suppose X, Y are linear spaces over a field F, and S is a set.

a. With the operations defined above, Y^S is a linear space over F; $L(X;Y)$ is a subspace of Y^X.

Suppose X has dimension n and Y has dimension m.

b. Choosing bases for X, Y, $[0_{ij}] = 0$, $[(f+g)_{ij}] = [f_{ij}] + [g_{ij}]$, and $[(cf)_{ij}] = [cf_{ij}]$.

d. $L(X;Y)$ has dimension mn.

Proof: Exercise 1. QED.

Lemma 2. Suppose X, Y are linear spaces over a field F, $U \subseteq X$ is an open subspace, $f, g \in Y^U$, and $x_0 \in U$.

a. $\lim_{x\to x_0}(f+g)(x) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$

b. $\lim_{x\to x_0}(cf)(x) = c \cdot \lim_{x\to x_0} f(x)$.

Suppose $F = \mathcal{R}$, and $Y = \mathcal{R}$. The operations $\cdot$ and $/$ are defined "pointwise", i.e., $(f \cdot g)(x) = f(x) \cdot g(x)$ and $(f/g)(x) = f(x)/g(x)$ provided $g(x) \neq 0$ for $x \in U$.

c. $\lim_{x\to x_0}(f \cdot g)(x) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$

d. $\lim_{x\to x_0}(f/g)(x) = \lim_{x\to x_0} f(x)/\lim_{x\to x_0} g(x)$

Proof: Exercise 2. QED.

The preceding lemma can be strengthened slightly, in that X can be any metric space. In part d, note that if g is continuous then $\{x : g(x) \neq$

$0\}$ is open (for X a topological space). As an immediate corollary of the lemma and theorem 2.6, if f and g are continuous it x_0, then $f+g$, cf, $f \cdot g$, and f/g are. Suitably modified, the theorem holds for infinite sequences.

Lemma 3. If $f \in L(X;Y)$ then $|f(x)| \leq \sqrt{mn}M|x|$, where $M = \max_{ij}\{|f_{ij}|\}$. It follows that f is continuous.

Proof: $|f(x)| = \sqrt{\sum_i(\sum_j f_{ij}x_j)^2} \leq \sqrt{m}\sqrt{n}M|x|_{\max}$, and $|x|_{\max} \leq |x|$. For the proof of the second claim, see the additional material at the end of the chapter. QED.

For theorem 4 and 5 below, suppose U is an open subset of $\mathcal{R}^n$, $x \in U$, and $f : U \mapsto \mathcal{R}^m$; for theorem 6 suppose $g : U \mapsto \mathcal{R}^m$ also.

Theorem 4. If the derivative of f at x exists then it is unique.

Proof: Suppose ϕ_1 and ϕ_2 are linear functions satisfying the requirement, and let $\psi = \phi_1 - \phi_2$. It is easily seen that $\lim_{\xi\to 0} |\psi(\xi)|/|\xi| = 0$, whence $|\psi(\xi)| < \epsilon|\xi|$ for any $\epsilon > 0$ and $\xi \neq 0$; fixing ξ and letting ϵ go to 0, it follows that $\phi(\xi) = 0$. QED.

If f is differentiable at x the notation $f'(x)$ is used to denote the value of the derivative.

Theorem 5. If f is differentiable at x then f is continuous at x.

Proof: $|f(x+\xi) - f(x)| \leq |f'(x)\xi| + |E|$. The first term is less than $b|\xi|$ for some $b > 0$ (lemma 3), which approaches 0 as ξ does. $|E|/|\xi|$ approaches 0 by definition; it follows that $|E|$ does. QED.

Theorem 6. Suppose f, g are differentiable at x.

a. $f+g$ is differentiable at x, and $(f+g)'(x) = f'(x) + g'(x)$.
b. cf is differentiable at x, and $(cf)'(x) = cf'(x)$.

Suppose $m = 1$.

c. $f \cdot g$ is differentiable at x, and $(f \cdot g)'(x) = f'(x)g(x) + g'(x)f(x)$.
d. Provided $g(x) \neq 0$, f/g is differentiable at x, and $(f/g)'(x) = (g(x)f'(x) - f(x)g'(x))/(g(x)^2)$.

Proof: Write $f(x+\xi) = f(x) + f'(x)(\xi) + E_f$, and $g(x+\xi) = g(x) + g'(x)(\xi) + E_g$, Adding these equations, $(f+g)(x+\xi) = (f+g)(x) + f'(x)(\xi) + g'(x)(\xi) + E_f + E_g$. Part a follows (using lemma 2). Multiplying the first equation by c, $(cf)(x+\xi) = (cf)(x) + cf'(x)(\xi) + cE_f$. Part b follows. Multiplying the two equations, $(fg)(x+\xi) = (fg)(x) + g(x)f'(x)(\xi) + f(x)g'(x)(\xi) + E$ where E consists of six "lower order" terms. One of these is $f'(x)(\xi)g'(x)(\xi)$; using lemma 3 it follows that this term goes to zero faster than ξ. The remaining five terms all involve E_f or E_g, multiplied by a factor either independent of ξ, or going to 0. Part c follows. Let $E_3 = g(x)E_f - f(x)E_g$; then

E_3 approaches 0 faster than ξ. Let $E_2 = E_3/(g(x+\xi)g(x))$; then E_2 approaches 0 faster than ξ. Further, $E_2 = f(x+\xi)/g(x+\xi) - f(x)/g(x) - (g(x)f'(x)(\xi) - f(x)g'(x)(\xi))/(g(x+\xi)g(x))$. Letting $E_1 = f(x+\xi)/g(x+\xi) - f(x)/g(x) - (g(x)f'(x)(\xi) - f(x)g'(x)(\xi))/(g(x)^2)$, it follows that E_1 approaches 0 faster than ξ. This proves part d. QED.

Parts a and b of the above theorem may be summarized as stating that the derivative operator is a "linear operator". The next theorem states another important property of the derivative operator, namely the "chain rule", which specifies the derivative of the composition of two functions.

Theorem 7. Suppose U is an open subset of $\mathcal{R}^n$, $f : U \mapsto \mathcal{R}^m$, V is an open subset of $\mathcal{R}^m$ with $f[U] \subseteq V$, $g : V \mapsto \mathcal{R}^l$, $x \in U$, f is differentiable at x, and g is differentiable at $f(x)$. Then $g \circ f$ is differentiable at x, and $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$.

Proof: It is convenient to write the error terms as $E_f = f(x') - f(x) - f'(x)(x'-x)$ and $E_g = g(y') - g(y) - g'(y)(y'-y)$, so that $\lim_{x' \to x} |E_f|/|x'-x| = 0$ and $\lim_{y' \to y} |E_g|/|y'-y| = 0$ where $y = f(x)$. Let $E_1 = g(f(x')) - g(f(x)) - g'(f(x))(f(x') - f(x))$. By the requirement on E_g, using theorem 5 and theorem 2.9, it follows that $\lim_{x' \to x} |E_1|/|x'-x| = 0$. Let $E_2 = g'(f(x))(f(x') - f(x) - f'(x)(x'-x))$. By the requirement on E_f, using lemma 3 and theorem 2.9, it follows that $\lim_{x' \to x} |E_2|/|x'-x| = 0$. Using lemma 2 it follows that $\lim_{x' \to x} |E|/|x'-x| = 0$ where $E = E_1 + E_2 = (g \circ f)(x') - (g \circ f)(x) - g'(f(x))(f'(x)(x'-x))$. QED.

$L(\mathcal{R}^n;\mathcal{R}^m)$ is, as observed in lemma 1, an mn-dimensional linear space over $\mathcal{R}$. It is essentially the same as $\mathcal{R}^{mn}$, and a matrix $[M_{ij}]$ may be equipped with the Euclidean norm $\sqrt{\sum_{ij} M_{ij}^2}$, making it a normed linear space. See the additional material for further remarks on matrix norms.

If f is differentiable at every point x of its domain U, then f is said to be differentiable. In this case f' may be considered as a function from U to $L(\mathcal{R}^n;\mathcal{R}^m)$. The function f' may or may not be continuous (see the additional material for further remarks). If it is continuous at $x \in U$ then f is said to be C_1 at x; if f is C_1 at each $x \in U$ it is said to be C_1. If f is C_1 at x and g is C_1 at $f(x)$ then since $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$, $g \circ f$ is C_1 at x. Various other properties of this notion will be seen below.

If $f \in L(\mathcal{R}^n;\mathcal{R}^m)$, then $f(x+\xi) - f(x) - f(\xi) = 0$, and it follows that at every point $x \in \mathcal{R}^n$, f is differentiable and $f'(x) = f$. In fact, f is C_1: f' is a constant function, and as is readily verified the derivative of a constant function is the 0 function, so f' is continuous.

In the case $n = 1$, there is a canonical bijective correspondence between $\mathcal{R}^m$ and $L(\mathcal{R};\mathcal{R}^m)$, where the vector $u \in \mathcal{R}^m$ corresponds to the function $t \mapsto tu$. Via this correspondence, f' may be considered to be a function from $\mathcal{R}$ to $\mathcal{R}^m$. Confusion is possible, and the version of f' intended might require clarification. For example, if $f(t) = tu$, then the function $f'(t)$ is $\xi \mapsto u\xi$, and the vector $f'(t)$ is u. In the case $m = 1$ also, the real number $f'(t)$ is the slope of the straight line in the plane, which contains the point $\langle t, f(t)\rangle$, and which "best approximates", or is "tangent to", the curve $\{\langle w, f(w)\rangle : w \in U\}$ (the graph of f) at the point $\langle t, f(t)\rangle$.

Lemma 8. If U is an open subset of $\mathcal{R}$ and $f : U \mapsto \mathcal{R}^m$ then $f'(t)$ exists iff $\lim_{\xi\to 0}(f(t + \xi) - f(t))/\xi$ exists, and in this case if u is the limit then $f'(t)(\xi) = u$.

Proof: This follows because $\lim_{\xi\to 0}(f(t + \xi) - f(t))/\xi = u$ iff $\lim_{\xi\to 0}(f(t+\xi)-f(t)-u\xi)/\xi = 0$ iff $\lim_{\xi\to 0}(f(t+\xi)-f(t)-u\xi)/|\xi| = 0$. QED.

Given $x \in \mathcal{R}^n$ and a "direction vector" $u \in \mathcal{R}^n$, let $h : \mathcal{R} \mapsto \mathcal{R}^n$ be the function where $h(t) = x + tu$. It is easily seen that h is everywhere differentiable, and $h'(t)(\xi) = u$; indeed h is C_1 everywhere. Using the chain rule and other facts given above, if $f : \mathcal{R}^n \mapsto \mathcal{R}^m$ is differentiable at x then $(f \circ h)'(0)$ exists and equals the function $\xi \mapsto \xi f'(x)(u)$. This expression is known as the "directional derivative" (of f at x in the direction u). By lemma 8, $f'(x)(u) = \lim_{\xi\to 0}(f(x + \xi u) - f(x))/\xi$; this vector is also called the directional derivative.

Let π_i^n denote the function from $\mathcal{R}_n$ to $\mathcal{R}$, which maps $\langle x_1, \ldots, x_n\rangle$ to x_i. Such a map is called a projection map. When n is clear, π_i may be written. Note that for $x \in \mathcal{R}^n$, $x = \langle \pi_1(x), \ldots, \pi_n(x)\rangle$. π_i is readily seen to be a linear transformation, whence it is C_1.

If $f : \mathcal{R}^n \mapsto \mathcal{R}^m$, $f(x) = \langle(\pi_1 \circ f)(x), \ldots, (\pi_m \circ f)((x)\rangle$; the function $\pi_i \circ f$ maps $\mathcal{R}^n$ to $\mathcal{R}$, and may be called the "ith component" of f.

When there is no danger of confusion, the notation f_i may be used for $\pi_i \circ f$. Corresponding to the function f from vectors to vectors is the "vector of scalar functions" $\langle f_1, \ldots, f_n\rangle$. Properties of interest of this correspondence include the following.

Theorem 9. Suppose U is an open subset of $\mathcal{R}^n$, $f : U \mapsto \mathcal{R}^m$, $x \in U$, and $f_i = \pi_i \circ f$ for $1 \le i \le m$.

a. f is continuous at x iff f_i is continuous at x for $1 \le i \le m$.
b. f is differentiable at x iff f_i is differentiable at x for $1 \le i \le m$, in which case $f_i'(x) = \pi_i \circ f'(x)$.
c. f is C_1 at x iff f_i is C_1 at x for $1 \le i \le m$.

Proof: π_i is linear, whence by facts already noted is C_1. It follows that if f is continuous then f_i is; if f is differentiable than f_i is, and

$f_i'(x) = \pi_i'(f(x)) \circ f'(x) = \pi_i \circ f'$; and if f is C_1 than f_i is. Suppose the f_i are all continuous, and $V \subseteq \mathcal{R}^m$. Let W be the Cartesian product $f_1^{-1}[V] \times \cdots \times f^{-1}[V]$. It is readily seen that this is open, using the fact that if $|w_i - x_i| < \delta_i$, $\delta = \min\{\delta_i\}$, and $\sqrt{\sum_i |w_i - x_i|^2} < \delta$, then $|w_i - x_i| < \delta \leq \delta_i$. (this argument can be generalized to show that the equipping the product of metric spaces with the "Euclidean" metric yields the product topology). Suppose the f_i are all differentiable at x. Let $|d_i = f_i(x+\xi) - f(x) - f_i'(x)(\xi)|$ and $|d = f(x+\xi) - f(x) - f'(x))(\xi)|$. Using the fact that $f_i' = (f')_i$, it follows that $d \leq \sqrt{\sum_i d_i^2} \leq \sqrt{n}\{\max |d_i|\}$, and then that f is differentiable at x. Suppose the f_i are all differentiable at x. Using the fact that $f_i' = (f')_i$, and the argument of part a, it follows that f is C_1 at x. QED.

Suppose $U \subseteq \mathcal{R}^m$ is an open subset, $f : U \mapsto \mathcal{R}$, and $x \in U$. If the directional derivative of f at x in the direction e_j exists, let $\partial f/\partial x_j$ denote it. This linear function from $\mathcal{R}$ to $\mathcal{R}$, or the real number corresponding to it, is called the partial derivative of f with respect to x_j. By lemma 8 the real number equals $\lim_{\xi \to 0}(f(x+\xi e_j) - f(x))/\xi$. The notion is sometimes extended to functions $f : U \mapsto \mathcal{R}^m$ "componentwise", i.e., the vector $\langle \partial f_1/\partial x_j, \ldots, \partial f_m/\partial x_j \rangle$ considered; but in basic calculus this is more often omitted and the quantities $\partial f_i/\partial x_j$ referred to directly.

Lemma 10 following will be needed for the proof of theorem 11, and has many other uses in vector calculus.

Lemma 10. Suppose $[l,u]$ is a closed interval in $\mathcal{R}$, $p : [l,u] \mapsto \mathcal{R}^n$ is continuous, and for $t \in (l,u)$ $p'(t)$ exists and $|p'(t)| \leq M$. Then $|p(u) - p(l)| \leq M(u-l)$.

Proof: Suppose $l < l' < u' < u$, and $\epsilon > 0$. Suppose $l' \leq v < u'$, and $|p(t) - p(l')| \leq (M+\epsilon)(t-l')$ for $t \in [l',v]$. There is a $\delta > 0$ such that $v + \delta \leq u'$ and for $0 \leq s \leq \delta$, $|p(v+s) - p(v) - p'(v)(s)| < \epsilon s$, whence $|p(v+s) - p(v)| \leq (M+\epsilon)s$, whence $|p(t) - p(l')| \leq (M+\epsilon)(t-l')$ for $t \in [l', v+\delta]$. It follow that $|p(t) - p(l')| \leq (M+\epsilon)(t-l')$ for $t \in [l',u']$. Since ϵ was arbitrary, $|p(t) - p(l')| \leq M(t-l')$ for $t \in [l',u']$. By continuity $|p(t) - p(l)| \leq M(t-l)$ for $t \in [l,u]$. QED.

Given $y, z \in \mathcal{R}^n$, let $p_{yz} : \mathcal{R} \mapsto \mathcal{R}^n$ be the function where $p_{yz}(t) = y + t(z-y)$. $\text{Ran}(p_{yz})$ is the "line segment" from y to z. A subset $S \subseteq \mathcal{R}^n$ is said to be convex if whenever $y, z \in S$, $\text{Ran}(p_{yz}) \subseteq S$. It is readily verified that an open ball is a convex subset (write $(1-t)y + tz$ for $y + t(z-y)$). This fact will be used in the proof of the following theorem.

The notation $\partial f_i/\partial x_j$ is unwieldy when the partial derivative, as a function of the position x, is to be considered. In the following theorem, the notation $f_i^{\partial j}$ will be introduced for this function, where $f_i^{\partial j}(x)$ is the real number, so that $f_i^{\partial j} : \mathcal{R} \mapsto \mathcal{R}$. Many authors use $D_i f$ for this.

Theorem 11. Suppose U is an open subset of $\mathcal{R}^n$, $f : U \mapsto \mathcal{R}^m$, and $x \in U$. If f is differentiable at x let $[f'(x)_{ij}]$ denote the matrix of $f'(x)$ with respect to the bases of standard unit vectors for $\mathcal{R}^n$ and $\mathcal{R}^m$.

a. If f is differentiable at x then $f'(x)_{ij} = f_i^{\partial j}(x)$.

b. If f is C_1 at x then $f_i^{\partial j}$ is C_1 at x.

c. If $f_i^{\partial j}$ is C_1 at x for all i, j then f is C_1 at x.

Proof: For any $h \in L(\mathcal{R}^n; \mathcal{R}^m)$ it is readily seen by composing both sides of the equation $h(e_j) = \sum_i h_{ij} i_i$ with π_i, that $h_{ij} = \pi_i(h(e_j))$. In particular, $f'(x)_{ij} = \pi_i(f'(x)(e_i))$. Parts a and b follow by theorem 9 and facts noted above about directional derivatives.

For part c, by theorem 9 it suffices to consider each f_i. Given $\xi \in \mathcal{R}^n$, let $\xi_{\restriction j}$ have components ξ_l if $l \leq j$, else 0, where $j = 0$ is allowed. Then $f_i(x + \xi) - f(x) - \sum_j f_i^{\partial j}(x)\xi_j = \sum_j (A_j + B_j)$ where $A_j = f(x + \xi_{\restriction j}) - f(x + \xi_{\restriction (j-1)}) - f_i^{\partial j}(x + \xi_{\restriction (j-1)})\xi_j$ and $B_j = (f_i^{\partial j}(x) - f_i^{\partial j}(x + \xi_{\restriction (j-1)})\xi_j$.

Given ϵ, there is a δ_1 such that $f_i^{\partial j}(x) - f_i^{\partial j}(x + \xi_{\restriction (j-1)}$ for $|\xi_j| < \delta$, by the assumption that $f_i^{\partial j}$ is continuous. Thus, $B_j/|\xi|$ approaches 0.

Now, let e_j^s be e_j if $\xi_j \geq 0$, else $-e_j$, and let $g(\zeta) = f(x + \xi_{\restriction (j-1)} + \zeta e_j^s) - f^{\partial j}(x + \xi_{\restriction (j-1)})\zeta$ for $0 \leq \zeta \leq 1$. It is easily seen that (the scalar) $g'(\zeta)$ equals $f^{\partial j}(x + \xi_{\restriction (j-1)} + \zeta e_j^s) - f^{\partial j}(x + \xi_{\restriction (j-1)})$. Given ϵ, by continuity of $f^{\partial j}$, δ can be chosen so that $|f(x') - f(x)| < \epsilon$ for $x' \in B_{x\delta}$, whence $|g'(\zeta)| < 2\epsilon$ for $\zeta \in [0, 1]$. Since $A_j = g(1) - g(0)$, by lemma 10, $|A_j| < 2|\xi_j|\epsilon$. Thus, $A_j/|\xi|$ approaches 0.

Altogether, this shows that f is differentiable at x. By part a and the hypothesis of part c, the components of the matrix $[f'(x)_{ij}]$ are all continuous as functions of x, whence by theorem 9.a $[f_{ij}]$ is. QED.

It is not true that if $\partial f_i/\partial x_j$ exists at x for all i, j then f is differentiable at x. Exercise 3 gives an example.

The notion of a C_1 function can be generalized, to that of a C_r function for $r \geq 1$. For $f : U \mapsto \mathcal{R}^m$ where U is an open subset of $\mathcal{R}^n$, let $f^{(0)}$ denote f; inductively if $f^{(r)}$ is differentiable let $f^{(r+1)}$ denote $(f^{(r)})'$. Letting $Y^{(0)}$ denote $\mathcal{R}^m$, and inductively letting $Y^{(r+1)}$ denote $L(\mathcal{R}^n; Y^{(r)})$, $f^{(r)} : U \mapsto Y^{(r)}$. If $f^{(r)}$ is defined, f is said to be r-times differentiable; if $f^{(r)}$ is continuous f is said to be C_r. The case $r = 0$ is included; a function f is C_0 iff it is continuous.

Additional material.

A linear transformation $f : X \mapsto Y$ between normed linear spaces over $\mathcal{R}$ is said to be bounded iff there is positive real number b such that $|f(x)| \leq b|x|$. By lemma 3 if X, Y are finite dimensional, then

f is bounded. Although examples would take us too far afield (see [wikiUnbO]), it is a fact that between infinite dimensional spaces, a linear transformation need not be bounded.

Theorem A1. Suppose $f : X \mapsto Y$ is a linear transformation between normed linear spaces over $\mathcal{R}$. Then f is continuous iff f is bounded.

Proof: Suppose $|f(x)| \leq b|x|$; then if $|x| < \epsilon/b$ then $|f(x)| < \epsilon$. Suppose $\forall\epsilon\exists\delta\forall x(|x| < \delta \Rightarrow |f(x)| < \epsilon)$. Let $\epsilon = 1$ and choose δ. Then $\forall c > 0\forall x(c|x| < \delta \Rightarrow c|f(x)| < 1)$. If $x \neq 0$ let $c = \delta/(2|x|)$; then $|f(x)| < (2/\delta)|x|$. Thus, for any x $|f(x)| \leq (2/\delta)|x|$. QED.

It is readily verified that the bounded linear operators are a subspace of the space of all linear operators. Indeed, it can be made into a normed linear space; but we consider this only for $L(\mathcal{R}^n; \mathcal{R}^m)$. In this case, by considering a matrix as a vector of length mn, three equivalent norms have already been defined, the Euclidean norm, the max norm, and the sum norm.

Sections 2.3.1 and 2.3.2 of [GvL] discuss some further matrix norms, which are equivalent to the above, and have properties of interest in various branches of mathematics and numerical analysis. Of particular interest is the value $\sup\{|f(x)| : |x| = 1\}$. This will be shown to be a norm in chapter 13. It goes by various names. In [GvL] it is called the "2-norm"; other names include the "strong norm" and the "operator norm".

The converse of theorem 5 does not hold. For example there are functions $f : \mathcal{R} \mapsto \mathcal{R}$ which are continuous everywhere but differentiable nowhere. Such an example will not be given here; see [wikiWeiF].

Exercises.

1. Prove lemma 1.

2. Prove lemma 2. Hint: Given ϵ, choose ϵ_f and ϵ_g suitably, then choose δ_f and δ_g, then let $\delta = \min\{\delta_f, \delta_g\}$. For part a let $\epsilon_f = \epsilon_g = \epsilon/2$. For part b let $\epsilon_f = \epsilon/|c|$. For part c let $\epsilon_f = \epsilon_g = (1/3)\min\{\epsilon, 1\}\min\{1, 1/|f(x_0)|, 1/|g(x_0)|\}$; use the identity $u'v' - uv = (u'-u)(v'-v)+u(v'-v)+v(u'-u)$. For part d, let $\epsilon_f = (|g(x_0)|/4)\epsilon$ and $\epsilon_g = \min\{|g(x_0)|/2, |g(x_0)|^2/(4|f(x_0)|)\}\epsilon$. Then $|(f(x)/g(x) - f(x_0)/g(x_0)| \leq |f(x)/g(x) - f(x_0)/g(x)| + |f(x_0)/g(x) - f(x_0)/g(x_0)| = (1/|g(x)|)|f(x) - f(x_0)| + |f(x_0)||g(x_0) - g(x)|/(|g(x_0)||g(x)|)$; and if $|x - x_0| < \delta$ then $|g(x)| \geq |g(x_0)|/2$.

3. Let $f : \mathcal{R}^2 \mapsto \mathcal{R}$ be the function where $f(0, 0) = 0$, and $f(x, y) = (xy)/(x^2 + y^2)$ otherwise.

a. Show that if $\langle x_0, y_0\rangle \neq \langle 0, 0\rangle$ then $\partial f/\partial x$ and $\partial f/\partial y$ exist at the point $\langle x_0, y_0\rangle$, Hint: $x^2 + y^2 \neq 0$, so theorem 6 can be used.

b. Show that at $\langle 0,0\rangle$, $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$. Hint: $f(x,0) = f(0,y) = 0$.
c. Show that f is not continuous at $\langle 0,0\rangle$. Hint: If $y = x$ then $f(x,y) = 1/2$.
d. Conclude that f is not differentiable at $\langle 0,0\rangle$. Hint: This follows by theorem 5.

6. Topology.

Topology is the branch of mathematics concerned with topological spaces. It has already been seen how the study of Euclidean space may be organized using basic concepts from topology. Further study will benefit from further facts from topology; these will be covered in this chapter.

If X is a topological space, a subset $K \subseteq X$ is called closed if K^c is open. It is easily seen that a subset $K \subseteq X$ is closed, iff for every $x \notin K$, there is an open set U with $x \in U$ and $U \cap K = \emptyset$.

Lemma 1. Suppose X is a topological space, $Y \subseteq X$ is a subspace, and $S \subseteq Y$.

a. If S is an open subset of X then S is an open subset of Y.
b. If Y is open then S is an open subset of X iff S is an open subset of Y.
c. If S is a closed subset of X then S is a closed subset of Y.
d. If Y is closed then S is a closed subset of X iff S is a closed subset of Y.

Proof: Exercise 1. QED.

Since the union of any collection of open sets is open, for any subset $S \subseteq X$, there is a unique largest open set U, such that $U \subseteq S$. This subset is called the interior of S, and denoted S^{int}. By set theory and the definitions, the intersection of any collection of closed sets is closed. It follows that for any subset $S \subseteq X$, there is a unique smallest closed set K, such that $S \subseteq K$. This subset is called the closure of S, and denoted S^{cl}. Define the boundary of a set S to be $S^{\text{cl}} - S^{\text{int}}$; write S^{bd} for the boundary.

Lemma 2. Suppose X is a topological space and $S \subseteq X$.

a. $x \in S^{\text{int}}$ iff there is an open set U with $x \in U \subseteq S$.
b. $x \in S^{\text{cl}}$ iff any open set containing x intersects S.
c. x is a limit point of S iff x is a limit point of S^{int}.
d. $S^{\text{int}} = ((S^c)^{\text{cl}})^c$.
e. $x \in S^{\text{bd}}$ iff x is a limit point of both S and S^c.
f. $S^{\text{bd}} \cap S^{\text{int}} = \emptyset$ and $S^{\text{cl}} = S^{\text{bd}} \cup S^{\text{int}}$.
g. $S^{\text{bd}} = (S^c)^{\text{bd}}$.
h. S^{bd} is closed.

Proof: Exercise 2. QED.

In any metric space, a closed ball is a set of the form $\{w : |w-x| \leq \epsilon\}$. A closed ball B is readily seen to be closed: Given w with $d(w,x) > \epsilon$ there is an open ball with center w which is disjoint from B.

In $\mathcal{R}^n$, an open cell is a set of the form $\{x : l_i < x_i < u_i\}$ for real numbers $l_1 < u_1, \ldots, l_n < u_n$. An open cell C is readily seen to be open: Given $x \in C$ choose ϵ so that $(x_i - \epsilon, x_i + \epsilon) \subseteq (l_i, u_i)$ for all i; then $x \in B_{x\epsilon} \subseteq C$.

In $\mathcal{R}^n$, a closed cell is a set of the form $\{x : l_i \leq x_i \leq u_i\}$ for real numbers $l_1 < u_1, \ldots, l_n < u_n$. A closed cell C is readily seen to be closed: Given $x \notin C$ there is some i so that $x_i \notin [l_i, u_i]$, whence there is some ϵ such that $(x_i - \epsilon, x_i + \epsilon) \cap [l_i, u_i] = \emptyset$. It follows that $B_{x\epsilon} \cap C = \emptyset$.

Many authors allow $l_i \leq u_i$ in the definition of a closed cell. Closed cells where $u_i = l_i$ are said to be degenerate. Facts about closed cells often hold for these as well, but as degenerate cases, sometimes requiring additional comments in proofs.

Theorem 3. In $\mathcal{R}^n$, the closure of an open ball $B_{x,\epsilon}$ equals the corresponding closed ball $\{w : |w-x| \leq \epsilon\}$; and the interior of a closed ball equals the corresponding open ball. The same is true with "ball" replaced by "cell".

Proof: For the open ball $B_{x,\epsilon}$, it suffices to show that if $|w-x| = \epsilon$ then any open ball $B_{w,\delta}$ has nonempty intersection with $B_{x,\epsilon}$. Let $v = w - \delta'(w-x)$ where $\delta' < \delta, \epsilon$; it is easy to see that $v \in B_{w,\delta} \cap B_{x,\epsilon}$. For the closed ball $B = \{w : |w-x| \leq \epsilon\}$, it suffices to show that if $B_{w,\delta} \subseteq B$ then $w \in B_{x\epsilon}$. But if not, then $|w-x| = \epsilon$; let $v = w + \delta'(x-w)$ where $\delta' < \delta$; it is easy to see that $v \in B_{w,\delta} \cap B_{x,\epsilon}$. The case of a cell is left to exercise 3. QED.

The preceding fact for open balls need not hold in an arbitrary metric space; see exercise 4.

A cover of a topological space X is a collection C of subsets of X such that $X = \bigcup C$. If C' is a cover of X and $C' \subseteq C$ then C' is called a subcover of C. If every subset of C is open then C is called an open cover.

A topological space X is said to be compact if every open cover has a finite subcover. The notion of a compact space is a technical one, but one that has a long history (see [wikiCompact]), and has become indispensible. Examples of its use will be seen later.

A subset $S \subseteq X$ is said to be compact if it is compact as a subspace. This is a frequently encountered situation in topology, where a property of spaces may be generalized to subsets by considering the subset to be the subspace. Say that a collection C of subsets of X covers a subset $S \subseteq X$ if $S \subseteq \bigcup C$ (for example C is a cover iff it covers X). It is easily seen that S is compact iff for every collection of open subsets of X which covers S there is a finite subcollection which covers S.

A topological space X is said to be Hausdorff if for any two distinct points x and y there are open subsets U_x and U_y such that $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$. It is easily seen that a metric space is Hausdorff.

Lemma 4.

a. A closed subset K of a compact topological space X is compact.

b. A compact subset K of a Hausdorff topological space X is closed.

Proof: For part a, if C is a cover of K by open subsets of X, then $C \cup \{K^c\}$ is an open cover of X. Letting F be a finite subcover, $F - \{K^c\}$ is a finite cover of K. For part b, suppose $x \notin K$. For each $w \in K$ choose disjoint open sets U_w, V_w with $w \in U_w$, $x \in V_w$. There is a finite set of U_w covering K, and the intersection of the corresponding V_w contains x and is disjoint from the union of the U_w, hence from K. QED.

Compactness is defined for topological spaces. Completeness is a property of metric spaces (more generally of "uniform spaces", which will not be defined here). An infinite sequence $\langle x_n \rangle$ in a metric space is said to be a Cauchy sequence if for all real $\epsilon > 0$ there is a natural number N such that $|x_n - x_m| < \epsilon$ whenever $n, m > N$. A metric space is said to be complete if every Cauchy sequence converges to some limit.

Theorem 5. $\mathcal{R}^n$ is a complete metric space for any n.

Proof: Consider first the case $n = 1$. Let $\langle x_n \rangle$ be a Cauchy sequence in $\mathcal{R}$. The set of values taken on by the sequence is bounded above. Indeed, choose any $\epsilon > 0$, choose an N so that if $i, j \geq N$ then $|x_i - x_j| < \epsilon$, and consider $x_N + \epsilon$. This bounds above x_n for $x \geq N$, so an upper bound can be obtained by considering the maximum of this and the x_n for $n \leq N$. A similar argument shows that the set of values is bounded below. It is clear that $\inf\{x_k : k \geq n\}$ exists for each n. Letting b_n denote this value, it is clear that $\sup\{b_n\}$ exists. Let x denote this value; we claim that $\lim_{n \to \infty} x_n = x$. Given ϵ, we may successively choose $N \leq M \leq L$ so that $|x - b_N| < \epsilon/3$, $|b_N - x_M| < \epsilon/3$, and $|x_M - x_n| < \epsilon/3$ for $n \geq L$, which proves the claim.

Now suppose $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{R}^n$. Let x_{ki} be the ith component of x_k. For fixed i, these form a Cauchy sequence in $\mathcal{R}$, since $|x_{ki} - x_{mi}| \leq |x_k - x_m|$. Let y_i be the limit, and let $y = \langle y_1, \ldots, y_n \rangle$. Given $\epsilon > 0$, we may choose N so that $|x_{ki} - y_i| < \epsilon/\sqrt{n}$ for i from 1 to n, and hence $|x_k - y| < \epsilon$, whenever $k \geq N$. QED.

A subset $S \subseteq X$ of a metric space X is said to be bounded if there is some open ball U such that $S \subseteq U$. Bounded subsets of interest include the closed balls, and open and closed cells.

Lemma 6. A closed cell in $\mathcal{R}^n$ is compact.

Proof: Let C be the cell. Suppose $\{U_\alpha\}$ is an open cover of C with no finite subcover. Replace C by 2^n equal size closed cells; at least one of these must have no finite subcover. Continuing, a descending chain D of closed cells is obtained. The corners nearest the origin form a Cauchy sequence, and since as seen in theorem 5 $\mathcal{R}^n$ is complete, this has some limit x, which is clearly in all the closed cells of D. There is some U in the cover containing x. Let V be an open cell contained in U, with x at the center. As soon as the width of a closed cell in D is small enough, it will be contained in V. But this is a contradiction since the cell then has the finite subcover $\{U\}$, whereas it was chosen not to have a finite subcover. QED.

Theorem 7. A subset $S \subseteq \mathcal{R}^n$ is compact iff it is closed and bounded.

Proof: Suppose S is compact. By lemma 4.b, since $\mathcal{R}^n$ is Hausdorff, S is closed. S has a cover by open balls, whence S has a finite cover by open balls, and it follows easily that S is bounded. Suppose S is closed and bounded. Then there is a closed cell C such that $S \subseteq C$. By lemma 6, lemma 4.a, and lemma 1, S is compact. QED.

The next two theorems are two of the main uses of compactness. Historically, theorem 8.b supplied a missing step in a standard proof of the "fundamental theorem of algebra", that a polynomial of degree n has exactly n complex roots, counting multiplicities. See chapter 7 of [DowdAlg].

Theorem 8.

a. If X, Y are topological spaces, $f : X \mapsto Y$ is continuous, and X is compact, then $f[X]$ is compact.
b. A continuous function $f : X \mapsto \mathcal{R}$ from a compact topological space to $\mathcal{R}$ takes on a maximum and a minimum value.

Proof: For part a, if C is an open cover of $f[X]$ then $\{f^{-1}[O] : O \in C\}$ is an open cover of X. For part b, by part a and theorem 7, $f[X]$ is closed and bounded. Since it it bounded, it has an inf l; and since it is closed, $l \in f[X]$. Similarly $\sup f[X] \in f[X]$. QED.

For the next theorem, a definition is required. A function $f : X \mapsto Y$, where X, Y are metric spaces, is said to be uniformly continuous if for all $x \in X$ and $\epsilon > 0$ there is a δ such that $d(w, x) < \delta$ implies $d(f(w), f(x)) < \epsilon$. This differs from the definition of continuity in that for ordinary continuity, δ may depend on x.

Theorem 9. If X, Y are metric spaces, $f : X \mapsto Y$ is continuous, and X is compact, then f is uniformly continuous.

Proof: Given ϵ, for each $p \in X$ let δ_p be such that $d(p,q) < \delta_p$ implies $d(f(p), f(q)) < \epsilon/2$, and let $O_p = \{q : d(p,q) < \delta_p/2\}$. The O_p comprise an open cover of X, so there is a finite subcollection $\{O_{p_1}, \ldots, O_{p_n}\}$ which is an open cover. Writing δ_i for δ_{p_i}, let $\delta = \min\{\delta_1, \ldots, \delta_n\}/2$. Suppose $d(q,r) < \delta$; there is a p_i with $d(q,p_i) < \delta_i/2$, and $d(r,p_i) \leq d(q,r) + d(q,p_i) < \delta + \delta_i/2 \leq \delta_i$. Hence $d(f(q), f(p_i)) < \epsilon/2$ and $d(f(r), f(p_i)) < \epsilon/2$, and hence $d(f(q), f(r)) < \epsilon$. QED.

A subset $S \subseteq X$ of a topological space X is said to be a dense subset if $S^{\rm cl} = X$, or equivalently, any open subset of X has nonempty intersection with S. It is a fact of interest in basic calculus that $\mathcal{Q}$ is a dense subset of $\mathcal{R}$. Indeed, if $r \in U$ where $U \subseteq \mathcal{R}$ is open, then for any ϵ there is a rational q with $r - q < \epsilon$, and hence a rational q with $q \in U$.

Lemma 10. Suppose X, Y are topological spaces with Y Hausdorff, and $f, g : X \mapsto Y$ are continuous. Then $\{x \in X : f(x) \neq g(x)\}$ is open.

Proof: Suppose $f(x) \neq g(x)$, and let U_f, U_g be disjoint open subsets of Y such that $f(x) \in U_f$ and $g(x) \in U_g$. Then $W = f^{-1}[U_f] \cap g^{-1}[U_g]$ is open, $x \in W$, and if $w \in W$ then $f(w) \neq g(w)$. QED.

Corollary 11. Suppose X, Y are topological spaces with Y Hausdorff, $S \subseteq X$ is a dense subset, and $f : S \mapsto Y$. Then there is at most one continuous function $\bar{f} : X \mapsto Y$ extending f.

Proof: If f_1, f_2 were two extensions, then by the lemma $\{x \in X : f(x) \neq g(x)\}$ is open, hence intersects S, a contradiction. QED.

For an extension to exist, by theorem 2.1.c, f must be continuous. A sufficient condition will be given in theorem 13 below. Before giving this, a useful lemma will be proved.

Lemma 12. Suppose X, Y are metric spaces, $f : X \mapsto Y$ is uniformly continuous, and $\langle x_i \rangle$ is a Cauchy sequence in X. Then $\langle f(x_i \rangle)$ is a Cauchy sequence in Y.

Proof: Given $\epsilon > 0$, choose δ so that $d(f(u), f(v)) < \epsilon$ whenever $d(u,v) < \delta$. Then choose N so that if $i, j \geq N$ then $d(x_i, x_j) < \delta$. Then $d(f(x_i, f(x_j)) < \epsilon$ for $i, j \geq N$. QED.

Theorem 13. Suppose X, Y are metric spaces with Y complete, $S \subseteq X$ is a dense subset, $f : S \mapsto Y$ is uniformly continuous. Then there is a unique continuous function $\bar{f} : X \mapsto Y$ extending f, and $\bar{f}$ is uniformly continuous.

Proof: For each $x \in X$, and each $i \in \mathcal{N}$, choose $s_{xi} \in S$ with $d(s(i), x) < 1/(i+1)$; then $\langle s_{xi} \rangle$ converges to x, and a fortiori is a Cauchy sequence. By lemma 12, $\langle f(s_{xi}) \rangle$ is a Cauchy sequence in Y. Since Y is complete this sequence has some limit y. Let $\bar{f}(x) = y$.

If $x \in S$, it follows by the continuity of f that $\bar{f}(x) = f(x)$; thus, $\bar{f}$ is an extension of f.

Given ϵ, let δ_1 be such that for $s, s' \in S$, if $d(s,s') < \delta_1$ then $d(f(s), f(s')) < \epsilon$. Given $x \in S$, let N_x be large enough so that if $i \geq N_x$ then $d(s_{xi}, x) < \delta_1/3$ and $d(f(s_{xi}), \bar{f}(x)) < \epsilon/3$. Let $s_x = s_{x,N_x}$. If $d(x,y) < \delta_1/3$ then $d(s_x, s_y) \leq d(x, s_x) + d(x,y) + d(y, s_y) < \delta_1$, whence $d(\bar{f}(x), \bar{f}(y)) \leq d(\bar{f}(x), f(s_x)) + d(f(s_x), f(s_y)) + d(f(s_y, \bar{f}(y)) < \epsilon$. This shows that $\bar{f}$ is uniformly continuous.

Uniqueness of $\bar{f}$ follows by corollary 11. QED.

Corollary 14. Suppose $f : \mathcal{Q} \mapsto \mathcal{R}$ is continuous. Then there is a unique continuous function $\bar{f} : \mathcal{R} \mapsto \mathcal{R}$ extending f.

Proof: Suppose $x \in \mathcal{R}$, and choose $a, b \in \mathcal{Q}$ such that $a < x < b$. f is uniformly continuous in $[a,b]$, and $[a,b] \cap \mathcal{Q}$ is dense in $[a,b]$, so there is a function $\tilde{f} : [a,b] \mapsto \mathcal{R}$ which extends $f \restriction [a,b]$, and is continuous at x; let $\bar{f}(x) = \tilde{f}(x)$. Then $\bar{f}$ extends f and is continuous at each x. QED.

The notion of connectedness is useful in calculus, as corollary 17 below shows. A topological space is said to be connected if it is not the disjoint union of two nonempty open subsets.

Theorem 15. If $f : X \mapsto Y$ is continuous and X is connected then $f[X]$ is connected.

Proof: If $f[X]$ is not connected, let U_1 and U_2 be open subsets of Y such that $f[X]$ is the disjoint union of the nonempty sets $f[X] \cap U_1$ and $f[X] \cap U_2$. It follows that X is the disjoint union of the nonempty subsets $f^{-1}[U_1]$ and $f^{-1}[U_2]$. QED.

By an interval in $\mathcal{R}$ is meant a subset I such that if $a, b \in I$ and $a \leq b$ then $[a,b] \subseteq I$. It is readily verified that an interval is of one of seven types, namely,

(a,b), $[a,b)$, $(a,b]$, $[a,b]$, (∞,∞), $[a,\infty)$, $(\infty,b]$.

Theorem 16. A nonempty subset $S \subseteq \mathcal{R}$ is connected iff it is an interval.

Proof: If S is not an interval take $x < y < z$ with $x, z \in S$ and $y \notin S$; then S is the disjoint union of $S \cap (-\infty, y)$ and $S \cap (y, \infty)$.

If S is not connected take U, V disjoint open sets with $S \subseteq U \cup V$. Take $x \in S \cap U$, $z \in S \cap V$ where we may assume $x < z$. Let $y = \sup(U \cap [x,z])$.

There is an ϵ such that $B_{z\epsilon} \subseteq V$; then $U \cap B_{z\epsilon} = \emptyset$, so $y \neq z$, and since $y \leq z$, $y < z$. If $y \in U$ held then y could not be an upper bound for $U \cap [x,z]$, so $y \notin U$.

There is an ϵ such that $B_{x\epsilon} \subseteq U$; so $x < y$. If $y \in V$ held then y could not be the least upper bound for $U \cap [x,z]$, so $y \notin V$.

It follows that $y \notin S$, and S is not an interval. QED.

Corollary 17. Let $I \subseteq \mathcal{R}$ be an interval, and let $f : I \mapsto \mathcal{R}$ be continuous. Then $f[I]$ is an interval, and if I is a closed interval $[a, b]$ then $f[I]$ is a closed interval.

Proof: By theorem 16 I is connected, so by theorem 15 $f[I]$ is connected, so by theorem 16 $f[I]$ is an interval. If I is a closed interval then I is compact, so $f[I]$ is compact, hence closed and bounded, hence a closed interval. QED.

This corollary is often called the "intermediate value theorem". It has a corollary, which will be needed in chapter 13. If I is an interval say that f is

increasing if $a < b \Rightarrow f(a) < f(b)$,
decreasing if $a < b \Rightarrow f(a) > f(b)$,
nondecreasing if $a \leq b \Rightarrow f(a) \leq f(b)$, and
nonincreasing if $a \leq b \Rightarrow f(a) \geq f(b)$

for any $a, b \in I$.

Corollary 18. If I is an interval, and $f : I \mapsto \mathcal{R}$ is continuous and injective, then f is increasing or decreasing.

Proof: Exercise 6. QED.

A map $f : X \mapsto Y$ between topological spaces is said to be a homeomorphism if is is bijective, and its inverse f^{-1} is also continuous.

The product (Cartesian product) of topological spaces may be defined, by equipping the Cartesian product set with a topology. This topology is defined by declaring $\{U_1 \times \cdots \times U_n : U_i$ is an open subset of $X_i\}$ to be a base for the topology. It is easy to see that the U_i may be restricted to be basic open sets.

In the case of $\mathcal{R}^n$, the base for the product topology is the open cells. The space $\mathcal{R}^n \times \mathcal{R}^m$ is "essentially the same" as $\mathcal{R}^{n+m}$. The bijection $\langle\langle x_1, \ldots, x_n\rangle, \langle y_1, \ldots, y_m\rangle\rangle \mapsto \langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle$ is an isomorphism of vector spaces, which is also a homeomorphism. Similarly, $\mathcal{R}^n \times \mathcal{R}^m$ is isomorphic to $\mathcal{R}^m \times \mathcal{R}^n$ via a map which may be defined on the underlying sets, etc.

Additional material.

A version of theorem 7 may be given for metric spaces. A metric space is called totally bounded iff for every $\epsilon > 0$ there is a finite set S of points such that $\{B_{x\epsilon} : x \in S\}$ is a cover of X.

Theorem A1. A subset $S \subseteq X$ of a metric space X is compact iff it is complete and totally bounded.

For a proof, see theorem 17.26 of [DowdAlg].

The definition of the power function, begun in the additional material of chapter 2, will be completed here. The reader is assumed to be familiar with the binomial theorem; for convenience a brief treatment will be given.

For $0 \leq k \leq n$ let

$$\binom{n}{k} = \frac{n!}{k!(n-k)!};$$

these quantities are known as a binomial coefficients" It is an easy exercise in algebra to verify that for $n > 0$ and $1 \leq k \leq n$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Theorem A2 (Binomial theorem).

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof: The proof is by induction on n. For the basis, $n = 0$ and both sides are 1. For the induction step,

$$\begin{aligned}(x+y)^{n+1} &= x\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + y\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ &= x^{n+1} + \sum_{k=0}^{n-1} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1} + y^{n+1} \\ &= x^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k}\right) x^k y^{n+1-k} + y^{n+1}.\end{aligned}$$

Lemma A3. The function $q \mapsto x^q$ from $\mathcal{Q}$ to $\mathcal{R}$ is continuous.

Proof: First suppose $x > 1$. Given x, q_0, and ϵ, let $\epsilon' = \epsilon/x^{q_0}$, and choose n so that $(x-1)/\epsilon' < n$. Then by the binomial theorem (or by a direct argument), $x < 1 + n\epsilon' < (1+\epsilon')^n$, so $x^{1/n} < 1 + \epsilon'$. It follows that if $0 < q < 1/n$ then $x^q - 1 < \epsilon'$. Also, $x^{-1/n} > 1/(1+\epsilon') > 1 - \epsilon'$, and It follows that if $0 > q > -1/n$ then $x^q - 1 > \epsilon'$. The claim for $x = 1$ is trivial, since the function is the identically 1 function. The claim for $x < 1$ follows from the claim for $x > 1$, since $1/x$ is continuous for $x > 0$, and the composition of continuous functions is continuous. QED.

By corollary 14, the function $q \mapsto x^q$ may be uniquely extended to a continuous function $r \mapsto x^r$ from $\mathcal{R}$ to $\mathcal{R}$. Indeed, take any sequence $\langle q_i \rangle$

of rational numbers in some open interval containing x, which converges to x; then $x^r = \lim_{i\to\infty} x^{q_i}$. It is easy to show (exercise A1) that for $x > 1$, x^r may be alternativey defined to be $\sup\{x^q : q \in \mathcal{Q}, q < r\}$. Lemma 2.A1 holds for the function x^r with real exponents; the proof is left to exercise A2.

Exercises.

1. Prove lemma 1.
2. Prove lemma 2.
3. Prove theorem 3 for cells.
4. Construct a metric space in which theorem 3 for open balls fails to hold. Hint: Consider the discrete metric, where $d(x, y) = 1$ if $y \neq x$.
5. Prove that a closed interval in $\mathcal{R}$ is compact, without using completeness. Hint: Let C be an open cover of an interval $[a, b]$. Say that $x \in [a, b]$ is reachable if there is a finite subcollection $F \subseteq C$ such that $[a, x] \subseteq \bigcup F$. Let $c = \sup\{x \in [a, b] : x \text{ is reachable}\}$; clearly $c \in [a, b]$. There is a $U \in C$ such that $c \in U$. It follows that c is reachable. It also follows that $c < b$ is impossible.
6. Prove corollary 18. Hint: Suppose $a < b$. Consider the case $f(a) < f(b)$ and $c < a$. Then $f(a) < f(c) < f(b)$ is impossible by the intermediate value theorem; likewise $f(c) > f(b)$ is impossible.

A1. Show that for $x > 1$, x^r may be defined to be $\sup\{x^q : q \in \mathcal{Q}, q < r\}$. Hint: Given r, let $\langle q_i \rangle$ be such that $q_i < q_{i+1} < r$ and $r - q_i < 1/(i + 1)$. Then $\lim_i x^{q_i} = \sup\{x^{q_i}\}$.

A2. Show that lemma 1.A1 holds for real exponents. Hint: For part a, let f_1 be the left side, and f_2 the right side, of the equation. These functions are continuous, and agree on $\mathcal{Q} \times \mathcal{Q}$, which is a dense subset of $\mathcal{R} \times \mathcal{R}$.

7. Matrix properties.

Matrices enjoy a great variety of properties. Some such will be required later in the text, and will be covered here. Recall the following from chapter 4.

- A linear transformation $f : \mathcal{R}^n \mapsto \mathcal{R}^m$ is a function which preserves the vector space operations of addition and scalar multiplication.
- A linear transformation also preserves 0 and subtraction.
- The linear transformations are in bijective correspondence with the $m \times n$ matrices of real numbers.
- Composition of linear transformations corresponds to matrix multiplication.
- The identity linear transformation is represented by the identity matrix I.

Recall also from chapter 4 that an $n \times n$ matrix M has a two-sided inverse iff it has a left inverse iff it has a right inverse iff $\det(M) \neq 0$.

If this holds, M is said to be invertible, or nonsingular; otherwise it is singular. The set of invertible $n \times n$ matrices is closed under the operations of multiplication and inverse, and contains I. These operations satisfy the group axioms given in chapter 1 (exercise 1). This group is called the "general linear group"; it will be denoted GL_n.

Next, the definition of a multilinear function will be given; it is used in several topics. Suppose $V_1, \ldots, V_k, W$ are vector spaces over a field F. A function $f : V_1 \times \cdots \times V_n \mapsto W$ is said to be multilinear if, for $1 \leq i \leq k$,

$$f(x_1, \ldots, x_i + x_i', \ldots, x_k) = f(x_1, \ldots, x_i, \ldots, x_k) + f(x_1, \ldots, x_i', \ldots, x_k),$$

and

$$f(x_1, \ldots, cx_i, \ldots, x_k) = cf(x_1, \ldots, x_i, \ldots, x_k).$$

That is, f is multilinear if it is linear in each argument. If $W = F$ a multilinear function is called a multilinear form. If $k = 2$ f is said to be bilinear.

Recall from chapter 2 that the inner product of two vectors $x, y \in \mathcal{R}^n$ equals $\sum_{i=1}^n x_i y_i$.

Theorem 1. The inner product is a bilinear form. Also it is symmetric, that is, $x \cdot y = y \cdot x$.

Proof: Exercise 2. QED.

A rigid transformation of $\mathcal{R}^n$ is defined to be a function $f : \mathcal{R}^n \mapsto \mathcal{R}^n$ which preserves distance, that is, such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in \mathcal{R}^n$. A translation of $\mathcal{R}^n$ is defined to be a function of the form $x \mapsto x + t$ for some $t \in \mathcal{R}^n$.

Theorem 2.

a. A translation is a rigid transformation.
b. If f is a rigid transformation then there is a rigid transformation g preserving 0, and a $t \in \mathcal{R}^n$, such that $f(x) = g(x) + t$ for all $x \in \mathcal{R}^n$.
c. A rigid transformation preserving 0 is a linear transformation.
d. A linear transformation is rigid iff it preserves the norm, iff it preserves the inner product.

Proof: For part a, $|(x+t)-(y+t)| = |x-y|$. For part b, let $g(x) = f(x) - f(0)$ and $t = -f(0)$. For part c, let f be the transformation, and let x' denote $f(x)$, etc. The proof that $y = cx$ imples $y' = cx'$ is given in exercise 3. The proof that $z = x + y$ imples $z' = x' + y'$ is given in exercise 4. For part d, since $|x| = d(0, x)$ and $d(x, y) = |x - y|$, f preserves distance iff it preserves the norm. Since $|x| = \sqrt{x, x}$ and $x \cdot y = ((x + y) \cdot (x + y) - x \cdot x - y \cdot y)/2$, f preserves the norm iff it preserves the inner product. QED.

A linear transformation which preserves the inner product is called an orthogonal transformation. A set of vectors $S \subseteq \mathcal{R}^n$ is said to be orthogonal if $x \cdot y = 0$ for all distinct $x, y \in S$; and orthonormal if in addition $|x| = 1$ for all $x \in S$. Clearly, if S is orthogonal (resp. orthonormal) then $f[S]$ is orthogonal (resp. orthonormal).

Theorem 3. U is the matrix of an orthogonal transformation of $\mathcal{R}^n$ iff its columns are orthonormal.

Proof: Column j is $U(e_j)$ where e_j is the j-th standard unit vector. Since $\{e_j\}$ is an orthonormal set, if U is orthogonal then its columns are orthonormal. Conversely, $U(x) \cdot U(y) = \sum_{ij} x_i y_j U(e_i) \cdot U(e_j)$, so if the columns are orthonormal the right side is $x \cdot y$. QED.

It is readily verified that the set of orthogonal matrices contains I and is closed under the operations of matrix multiplication and inverse. It thus forms a subgroup of the general linear group.

Recall from chapter 4 the transpose operation M^T on a matrix M. The transpose operation obeys the identities $(M + N)^T = M^T + N^T$ and $(MN)^T = N^T M^T$; the proof is left to exercise 5.

Theorem 4. If M is an $n \times n$ matrix then $\det(M^T) = \det(M)$.

Proof: First, for $\sigma \in \mathrm{Sym}_n$, $\mathrm{sg}(\sigma)\,\mathrm{sg}(\sigma^{-1}) = 1$, so $\mathrm{sg}(\sigma) = \mathrm{sg}(\sigma^{-1})$. Second, $M^T_\sigma = M_{\sigma^{-1}}$. Third, $\sigma \mapsto \sigma^{-1}$ is bijective. It follows that $\sum_\sigma \mathrm{sg}(\sigma) M^T_\sigma = \det(M)$. QED.

By theorem 3, U is the matrix of an orthogonal transformation iff $U^T U = I$. In this case, by theorem 3.6, U^T equals U^{-1}, the two-sided inverse of U. Using theorem 4, $\det(U)^2 = 1$, whence $\det(U) = \pm 1$.

Given $\sigma \in \mathrm{Sym}_n$ let P^σ denote the $n \times n$ matrix where P^σ_{ij} equals 1 if $j = \sigma(i)$, else 0. The map $\sigma \mapsto P^\sigma$ is a bijection between Sym_n and the matrices whose columns (equivalently rows) are the standard unit vectors in some order; such matrices are called permutation matrices. The following are readily verified.

- $P^{\sigma\tau} = P^\tau P^\sigma$.
- $P^{\sigma^{-1}} = (P^\sigma)^{-1} = (P^\sigma)^T$.
- If $B = P^\sigma A$ then $B_{i\sigma(j)} = A_{ij}$ (i.e., row i gets moved to row $\sigma(i)$).
- If $B = A(P^\sigma)^T$ then $B_{\sigma(i)j} = A_{ij}$ (i.e., column j gets moved to column $\sigma(j)$).
- $\det(P^\sigma)$ equals $+1$ if σ is even and -1 if σ is odd. It follows that if the rows or columns of a matrix are permuted then the determinant is unchanged if the permutation is even, and multiplied by -1 if the permutation is odd.

The set of permutation matrices forms a subgroup of the general linear group. The map $\sigma \mapsto P^\sigma$ is in fact an isomorphism of groups.

An $n \times n$ matrix M is said to be diagonal if $M_{ij} = 0$ when $i \neq j$. These matrices are closed under matrix addition, matrix multiplication, scalar multiplication, and contain the identiry matrix. A diagonal matrix is invertible iff it's diagonal entries are all nonzero; the invertible diagonal matrices form a subgroup of the general linear group.

For $c \neq 0$ and k, l with $k \neq l$, let A^{ckl} be the $n \times n$ matrix whose i, j entry is 1 if $i = j$, c if $\langle i, j\rangle = \langle k, l\rangle$, and 0 otherwise. Such matrices will be called A-matrices; they are also called type 3 elementary matrices. It is readily verified that if M is an $m \times n$ matrix then MA^{ckl} is the matrix obtained from M by adding c times column k to column l; and for an $m \times m$ matrix A^{ckl}, $A^{ckl}M$ is the matrix obtained from M by adding c times row l to row k. It is readily verified that the inverse of A^{ckl} is $A^{-c,kl}$.

Theorem 5. An invertible $n \times n$ matrix M may be written as a product of permutation matrices, A-matrices, and a diagonal matrix.

Proof: If $n = 1$ the theorem is trivial. Otherwise, since M is invertible, there is a permutation matrix P such that, letting $M_1 = PM$, $M_{1,11} \neq 0$. There is then a product A of A-matrices such that, letting $M_2 = AM_1$, $M_{1,11}$ is unchanged and $M_{2,i1} = 0$ if $2 \leq i \leq n$. There is then a product B of A-matrices such that, letting $M_3 = M_2B$, the first column is unchanged and $M_{3,1j} = 0$ if $2 \leq j \leq n$. Let $\check{M}_3$ be M_3, with row and column 1 deleted. Calling matrices as in the statement of the theorem special, inductively, there are special $(n-1) \times (n-1)$ matrices $S_1, \cdots, S_r$, such that $\check{M}_3 = S_1 \cdots S_r$. For $1 \leq s \leq r$ let $\hat{S}_s$ denote S_s, with a 1 added in the upper left corner if S_s is a permutation matrix or A-matrix, or $M_{3,11}$ if S_s is the diagonal matrix; and 0 in the other entries of the first row and column. It is readily verified that $\hat{S}_s$ is a special $n \times n$ matrix (indeed of the same type as S_s), and that $M = P^{-1}A^{-1}\hat{S}_1 \cdots \hat{S}_r B^{-1}$. QED.

Exercises.

1. Prove that the invertible $n \times n$ matrices form a group under matrix multiplication.

2. Prove theorem 1.

3. Prove the scalar multiplication claim of theorem 2.c, by proving the following series of claims.

1. $|x + y| = |x| + |y|$ iff $x \cdot y = |x|\,|y|$ iff $x/|x| = y/|y|$. Hint: Refine the proof of lemma 1.
2. If $c > 1$ then $y = cx$ iff $|y| = c|x|$ and $|y| = |x| + |y - x|$. Hint: One direction is a straightforward computation using the hypothesis. For the converse, use claim 1 applied to $|x + (y - x)| = |x| + |y - x|$.
3. If $y = -x$ iff $|y| = |x|$ and $|x - y| = 2|x|$. Hint: Proceed as in claim 2, using claim 1 applied to $|x - y| = |x| + |-y|$.

4. If $c > 1$ then $y = cx$ iff $y' = cx'$. Hint: Use claim 2 in the forward direction, then the reverse.
5. If $0 < c < 1$ then $y = cx$ iff $y' = cx'$. Hint: Use claim 4 with x and y reversed.
6. If $c < 0$ then $y = cx$ iff $y' = cx'$. Hint: Negate both sides, and use claims 4 and 5, then claim 2.

4. Prove the addition claim of theorem 2.c, Hint: Provided y is not a scalar multiple of x, $z = x + y$ iff there is some w such that $d(x, w) = d(w, y)$, $d(x, y) = 2d(x, w)$, and $z = 2w$. Using exercise 2, the claim follows.

5. Show that $(M + N)^T = M^T + N^T$ and $(MN)^T = N^T M^T$.

6. Give an explicit representation of the 2×2 orthogonal matrices of determinant $+1$, and those of determinant -1. Give a geometric description of the corresponding rigid transformations.

7. Verify the facts stated above concerning permutation matrices.

8. Measure and integration.

In applications of vector calculus, various notions of measure and integral are encountered:

The "n volume" of a subset of $\mathcal{R}^n$.

The integral of a function $f : D \mapsto \mathcal{R}$, where $D \subseteq \mathcal{R}^n$.

If k is a "k-surface" in $\mathcal{R}^n$, its "k-volume".

The integral of a differential form over a k-surface in $\mathcal{R}^n$.

The situation is further complicated by the fact that that there are two notions of measure, Jordan and Lebesgue, and two corresponding notions of integral, Riemann and Lebesgue.

A measure μ on a space X is a function $\mu : A \mapsto \mathcal{R}$, where $A \subseteq \mathrm{Pow}(X)$, which satisfies some axioms. It is characteristic of measures that A need not be all of $\mathrm{Pow}(X)$, that is, only some subsets of X are assigned a measure. We begin by defining a simple "finitely additive" measure on $\mathcal{R}^n$.

In $\mathcal{R}^n$ define an H-cell to be a set of the form $\{x : l_i \leq x_i < u_i \text{ for all } i\}$ for real numbers $l_1 < u_1, \ldots, l_n < u_n$. A collection of subsets of a set X which is closed under pairwise difference and union is called a ring of sets. (As usual, this is an example of a "name collision" in mathemttics; the use of the term ring in this context is distinct from its use as given in chapter 1). Define a subset of $\mathcal{R}^n$ to be elementary if it is the union of a finite disjoint collection of H-cells.

Define $\nu(C)$ for an H-cell C to be $(u_1 - l_1) \cdots (u_n - l_n)$. Call a set of the form $\{x \in \mathcal{R}^n : x_i = c\}$ for some i and c a cutting hyperplane. Such a hyperplane partitions $\mathcal{R}^n$ into the sets $\{x \in \mathcal{R}^n : x_i < c\}$ and $\{x \in \mathcal{R}^n : x_i \geq c\}$. An H-cell C is partitioned into two parts C_1 and C_2 consisting of its intersection with these two sets (it is possible that one part is the whole H-cell and the other empty). The nonempty parts are H-cells, and as is readily verified $\nu(C) = \nu(C_1) + \nu(C_2)$, where $\nu(\emptyset)$ is defined to be 0. A bounding hyperplane of an H-cell $\{x : l_i \leq x_i < u_i\}$ is defined to be any cutting hyperplane $\{x : x_i = l_i\}$ or $\{x : x_i = u_i\}$.

Lemma 1.

a. The elementary sets form a ring of subsets of $\mathcal{R}^n$.

b. ν can be extended to the elementary sets so that if E is an elementary set then $\nu(E) = \sum_i \nu(C_i)$ where $\{C_i\}$ is any decomposition of E into disjoint H-cells.

c. If E and F are disjoint then $\nu(E \cup F) = \nu(E) + \nu(F)$.

Proof: For part a, partition all the H-cells in E and F by the bounding hyperplanes of the H-cells in E and F, and let H be the resulting collection of H-cells. It is readily seen that $E \cup F$ or $E - F$ is the union of a disjoint collection of H-cells from H. It is also readily seen that if $C = C_1 \cup \ldots \cup C_n$ where the C_i are disjoint then $\nu(C) = \nu(C_1) + \cdots + \nu(C_n)$. Part b follows, and also part c. QED.

Let $\mathcal{E}$ denote the ring of elementary sets. Define a set $S \subseteq \mathcal{R}^n$ to be Jordan measurable if $\inf\{\nu(E) : E \in \mathcal{E}, S \subseteq E\}$ equals $\sup\{\nu(E) : E \in \mathcal{E}, E \subseteq S\}$; in this case, let $\mu(S)$ denote the common value. Let $\mathcal{J}$ denote the Jordan measurable subsets.

For the following, the notion of a finitely additive measure is introduced. This is a function $\mu : A \mapsto \mathcal{R}$ where A is a ring of subsets of $\mathcal{R}^n$, such that $\mu(S) \geq 0$ for all $S \in A$, and if $S_1, \ldots, S_r$ is a finite collection of disjoint members of A then $\mu(\bigcup_i S_i) = \sum_i \mu(S_i)$. Note that ν is a finitely additive measure with domain $\mathcal{E}$.

Useful properties of a finitely additive measure μ on a ring of sets A which follow from the definition include the following.

- If $S_1, S_2 \in A$ and $S_1 \subseteq S_2$ then $\mu(S_1) \leq \mu(S_2)$ (monotonicity).
- If $S_1, S_2 \in A$ then $\mu(S_1 \cup S_2) \leq \mu(S_1) + \mu(S_2)$ (finite subadditivity).

The simple proofs are left to exercise 1.

Lemma 2. $S \in \mathcal{J}$ iff for every ϵ there are $E^I, E^O \in \mathcal{E}$ such that $E^I \subseteq S \subseteq E^O$ and $\nu(E^O) - \nu(E^I) < \epsilon$.

Proof: Suppose $S \in \mathcal{J}$. Let $m = \mu(S)$. There is an E^I with $E^I \subseteq S$ and $\nu(S) - m < \epsilon/2$; and an E^O with $S \subseteq E^O$ and $\nu(E^O) - \nu(S) < \epsilon/2$. Conversely, taking the values $\nu(E^I)$, $\nu(E^O)$ yields a Cauchy sequence converging to some m. Both the sequences $\nu(E^I)$ and $\nu(E^I)$ converge to m. QED.

Theorem 3. $\mathcal{J}$ is a ring of sets, μ is a finitely additive measure, $\mathcal{E} \subseteq \mathcal{J}$, and $\mu(S) = \nu(S)$ for $S \in \mathcal{E}$.

Proof: Given S_1, S_2, suppose that for $i = 1, 2$, $E_i^I \subseteq S_i \subseteq E_i^O$ and $\nu(E_i^O) - \nu(E_i^I) < \epsilon/2$. Then $E_1^I \cap E_2^I \subseteq S_1 \cap S_2 \subseteq E_1^O \cap E_2^O$. By set theory, $(E_1^O \cap E_2^O) - (E_1^I \cap E_2^I) \subseteq (E_1^O - E_1^I) - (E_2^O - E_2^I)$; it follows that $S_1 \cap S_2 \in \mathcal{J}$. Let C be a cell such that $S_1, S_2 \subseteq C$. It is easily seen that $C - S_i \in \mathcal{J}$, by replacing E^O by $C - E^I$ and E^I by $C - E^O$. It follows by set theory that $\mathcal{J}$ is a ring of sets. Suppose $S_1 \cap S_2 = \emptyset$. If $\nu(E_i^O)$ convrges to m_i then it may be seen that $\nu(E_1^O + E_2^O)$ converges to $m_1 + m_2$; similarly $\nu(E_1^I + E_2^I)$ converges to $m_1 + m_2$. It follows that μ is finitely additive. If $S \in \mathcal{E}$, then E^I and E^O may be taken as S, and the remaining claims follow. QED.

Theorem 4. If $S \in \mathcal{J}$ then $S^{\text{bd}} \in \mathcal{J}$ and $\mu(S^{\text{bd}}) = 0$, $S^{\text{int}} \in \mathcal{J}$ and $\mu(S^{\text{int}}) = \mu(S)$, and $S^{\text{cl}} \in \mathcal{J}$ and $\mu(S^{\text{cl}}) = \mu(S)$.

Proof: Given ϵ, choose $E^I, E^O \in \mathcal{E}$ such that $E^I \subseteq S \subseteq E^O$ and $\nu(E^O) - \nu(E^I) < \epsilon$. Let $D = (E^O)^{\text{cl}} - (E^I)^{\text{cl}}$; then $S^{\text{bd}} \subseteq D$ and $\mu(D) < \epsilon$. Thus there is an $F \in \mathcal{E}$ such that $S^{\text{bd}} \subseteq F$ and $\mu(F) < \epsilon$. Since ϵ was arbitrary the first claim follows. The second and third claims follow using theorem 3. QED.

A finitely additive measure μ on a ring of sets D is said to be translation invariant if, if f is the translation $x \mapsto x + t$ and $S \in D$, then $f[S] \in D$ and $\mu(f[S]) = \mu(S)$.

Theorem 5. The Jordan measure is translation invariant.

Proof: The claim for S is immediate if S is an H-cell. A translation preserves disjointness, and the claim follows for $S \in \mathcal{E}$. A translation preserves inclusion, and the claim follows for $S \in \mathcal{J}$. QED.

Theorem 6. If $S \in \mathcal{J}$ and f is an invertible linear transformation with matrix M then $f[S] \in \mathcal{J}$ and $\mu(f[S]) = |\det(M)|\mu(S)$.

Proof: By theorem 7.5, it suffices to show the claim when M is either a permutation matrix, an A-matrix, or a diagonal matrix. Further, as in the proof of theorem 5, it suffices to prove the claim for H-cells. A diagonal matrix or a permutation matrix M maps a closed cell to a closed cell; direct computation shows that the measure is multiplied by $|\det(M)|$. The claim follows for H-cells by theorem 5.3 and theorem 4. The claim for A-matrices is left as exercise 2. QED.

The Riemann integral will be defined for functions with the following properties.

1. $f : D \mapsto \mathcal{R}$ where D is a Jordan measurable subset of $\mathcal{R}^n$.
2. f is bounded on D, i.e., $f[D]$ is a bounded subset of $\mathcal{R}$.

A function with these properties will be called a $\mathcal{J}$-function. If in addition

3. f is nonnegative, i.e., $f(x) \geq 0$ for all $x \in D$,

the function will be said to be a $\mathcal{J}^{\geq}$-function. In the approach used here, it is convenient to define the Riemann integral first for $\mathcal{J}^{\geq}$-functions.

Other approaches define it directly for $\mathcal{J}$-functions; these approaches have other peculiarities, and they are all equivalent.

For a $\mathcal{J}^{\geq}$-function f let $V_f = \{\langle x_1, \ldots, x_n, x_{n+1}\rangle : \langle x_1, \ldots, x_n\rangle \in D$ and $0 \leq x_{n+1} \leq f(x_1, \ldots, x_n)\}$. The idea is, that f is integrable iff V_f is Jordan measurable; however a definition along more traditional lines will be given. For this, let $\mathcal{E}_b$ denote the elementary sets which are a disjoint union of H-cells with $l_{n+1} = 0$. For a set $E \in \mathcal{E}_b$, the set of $\langle x_1, \ldots, x_n\rangle$ such that $\langle x_1, \ldots, x_n, 0\rangle \in E$ will be referred to as the "base" of E; it may also desecribed as the projection of E onto the hyperplane $x_{n+1} = 0$. The value x_{n+1} may be called the "height".

Say that f is Riemann integrable iff $\inf\{\nu(E) : E \in \mathcal{E}_b, V_f \subseteq E\}$ equals $\sup\{\nu(E) : E \in \mathcal{E}_b, E \subseteq V_f\}$; in this case, let $\int_D f$ denote the common value.

Theorem 7. A $\mathcal{J}^{\geq}$-function f is Riemann integrable iff V_f is Jordan measurable, in which case $\int_D f = \mu(V_f)$.

Proof: Clearly $\inf\{\nu(E) : E \in \mathcal{E}_b, V_f \subseteq E\} \geq \inf\{\nu(E) : E \in \mathcal{E}, V_f \subseteq E\}$. In fact, equality holds, because given any $E \in \mathcal{E}$ with $V_f \subseteq E$ there is an $E' \in \mathcal{E}_b$ with $V_f \subseteq E' \subseteq E$. Similarly $\sup\{\nu(E) : E \in \mathcal{E}, E \subseteq V_f\} \geq \sup\{\nu(E) : E \in \mathcal{E}_b, E \subseteq V_f\}$. In fact, equality holds, because given any $E \in \mathcal{E}$ with $E \subseteq V_f$ there is an $E' \in \mathcal{E}_b$ with $E \subseteq E' \subseteq V_f$. Remaining details are left to exercise 3. QED.

In the case $D \in \mathcal{E}$, the sets E can further be restricted to have D as their base. This is a variation of the definition of the Riemann integral using "step functions".

For a $\mathcal{J}$-function, let c be large enough constant so that $f + c$ is a $\mathcal{J}^{\geq}$-function, and let $\int_D f = \int_D (f + c) - c\mu(D)$. It is easy to check that this value does not depend on c (some relevant facts may be found in the proof of the next theorem).

Theorem 8.

a. If f is the constant function $f(x) = 1$ then f is Riemann integrable, and $\int_D f = \mu(D)$.
b. If f and g are Riemann integrable functions with domain D, then $f + g$ is Riemann integrable, and $\int_D (f + g) = \int_D f + \int_D g$.
c. If f is a Riemann integrable function with domain D, then cf is Riemann integrable, and $\int_D (cf) = c \int_D f$.
d. If D is the disjoint union of Jordan measurable sets D_1 and D_2 then $\int_D f$ exists iff both $\int_{D_1} f$ and $\int_{D_2} f$ do, in which case $\int_D f = \int_{D_1} f + \int_{D_2} f$.
e. If f is a $\mathcal{J}$ function and $\mu(D) = 0$ then f is Riemann integrable, and $\int_D f = 0$.

Proof: Exercise 4. QED.

Claims b and c of the preceding theorem show that $\int_D$ is a linear operator on the vector space of J-functions with domain D. It also follows that if f and g are Riemann integrable functions with domain D, and $f \leq g$ (i.e., $f(x) \leq g(x)$ for all $x \in D$) then $\int_D f \leq \int_D g$. In particular, if $f \leq c$ then $\int_D f \leq c\mu(D)$. Finally, if f is a $\mathcal{J}$-function on a closed cell (resp. H-cell) D, and D' is the H-cell (resp. open cell) with the same l_i and u_i values, then the integral of f over D exists iff the integral over D' exists, in which case the values are the same.

Theorem 9. Suppose D is a closed cell, $f : D \mapsto \mathcal{R}$ is bounded, $N \subseteq D$, $N \in \mathcal{J}$, $\mu(N) = 0$, and f is continuous on $D - N$. Then f is Riemann integrable.

Proof: It suffices to consider consider f a $\mathcal{J}^{\geq}$ function; let M be such that $f(x) \leq M$ for $x \in D$. Let ϵ be given. Choose $E \in \mathcal{E}$ with $N \subseteq E^{\mathrm{int}}$ and $\mu(E) < \epsilon$. $D - E^{\mathrm{int}}$ is compact, so by theorem 5.8 f is uniformly continuous on $D - E^{\mathrm{int}}$. Letting D^H denote the H-cell corresponding to D, a partition of D^H can be chosen, which refines E, and such that for each H-cell B of the subdivision, if $B \subseteq D^H - E$ then there are l, u with $l < u$, $u - l < \epsilon$, and $l \leq f(x) \leq u$ for $x \in B$. It follows that there are elements $E^I, E^O \in \mathcal{E}_b$ with base D^H such that $E^I \subseteq V_{f \restriction D^H} \subseteq E^O$ and $\nu(E^O) - \nu(E^I) < \epsilon \cdot \nu(D) + \epsilon \cdot M$. QED.

This theorem can be strengthened; see the additional material.

Corollary 10. Suppose $D \in \mathcal{J}$, and $f : D \mapsto \mathcal{R}$ is bounded and continuous on D; then f is Riemann integrable.

Proof: Let C be a closed cell such that $D \subseteq C$. Extend f to C by setting $f(x) \neq 0$ if $x \in C - D$. Then f is continuous at x if $x \notin D^{\mathrm{bd}}$. QED.

Theorem 12 below (the fundamental theorem of calculus) gives a method for computing integrals in one dimension. The reader may be familiar with this theorem from introductory calculus; since it is of such great importance a proof is given here. The notation $\int_a^b f(x)dx$ will be used, presuming $a \leq b$, as an alternative to $\int_{[a,b]} f$.

The following lemma gives an alternative characterization of the Riemann integral (the one used by Riemann), which is sometimes useful in proofs. Suppose $f : [a, b] \mapsto \mathcal{R}$ is bounded. Define a sample σ of f to be a subdivision $a = x_0 < \cdots < x_n = b$, together with points y_i, $0 \leq i < n$, satisfying $x_i \leq y_i \leq x_{i+1}$. Define $I(\sigma) = \sum_{i=0}^{n-1} f(y_i)(x_{i+1} - x_i)$, and $M(\sigma) = \min\{x_{i+1} - x_i : 0 \leq i < n\}$ (this is often called the mesh).

Lemma 11. $\int_a^b f(x)dx = v$ iff

(*) for any $\epsilon > 0$ there is a $\delta > 0$ such that for any sample σ, $|v - I(\sigma)| < \epsilon$ whenever $M(\sigma) < \delta$.

Proof: Exercise 5. QED.

Theorem 12. Let f be integrable on $[a, b]$.

a. Let $F(x) = \int_a^x f(x)dx$ for $x \in (a, b)$; if f is continuous at x then F is differentiable at x and $F'(x) = f(x)$.

b. If $F' = f$ on some open interval containing $[a, b]$ then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof: For part a, the error term is as follows.

$$\begin{aligned} E_F(\Delta x) &= F(x + \Delta x) - F(x) - \Delta x f(x) \\ &= \int_a^{x+\Delta x} f(\xi)d\xi - \int_a^x f(\xi)d\xi - \Delta x f(x) \\ &= \int_x^{x+\Delta x} f(\xi)d\xi - \Delta x f(x) \end{aligned}$$

(if $\Delta x < 0$ the integration limits are reversed). Suppose f is continuous at x; given $\epsilon > 0$ there is a $\delta > 0$ such that $|f(\xi) - f(x)| < \epsilon/2$ if $|\xi - x| < \delta$. It is clear that that $|E_F(\Delta x)| < \epsilon|\Delta x|$ if $|\Delta x| < \delta$, which proves part a. Indeed, $f(\xi)$ is bounded below by $f(x) - \epsilon/2$ and above by $f(x) + \epsilon/2$. For part b, let $a = x_0 < \cdots < x_n = b$ be any subdivision of $[a, b]$. Let $y_i \in [x_i, x_{i+1}]$, $0 \leq i < n$, be such that $F(x_{i+1}) - F(x_i) = (x_{i+1} - x_i)f(y_i)$; then $F(b) - F(a) = \sum_{i=0}^{n-1}(x_{i+1} - x_i)f(y_i)$. Since the subdivision was arbitrary, by lemma 11 the right side must be $\int_a^b f(x)dx$. QED.

By the theorem, to compute $\int_a^b f(x)dx$ it suffices to find a function F, such that on some open interval containing $[a, b]$, $F' = f$. Such a function is called a primitive or antiderivative of f. A primitive is not unique; if C is any constant then $(F + C)' = F'$. On the other hand, if $(F_1 - F_2)'(x) = 0$ for $x \in (a, b)$, it is not difficult to show using basic calculus that $F_1(x) - F_2(x) = C$ for some constant C. The value selected for the arbitrary constant C is called the constant of integration.

The area of a half-circle of radius r is $\int_{-r}^r \sqrt{r^2 - x^2}dx$. This cannot yet be computed, because the antiderivative here is a "transcendental function". The next four chapters will give various facts needed to define the antiderivative, and the area of a circle will then be determined.

Volumes of solids can be computed by performing two successive one dimensional integrations. The next theorem is often usful in a theoretical justification. For this, some additional notation will be introduced. Suppose D is an H-cell, and $f : D \mapsto \mathcal{R}$ is nonnegative and bounded. Let $I_U = \inf\{\nu(E) : E \in \mathcal{E}_b,\ V_f \subseteq E,$ base of $E = D\}$; and $I_L = \sup\{\nu(E) : E \in \mathcal{E}_b,\ E \subseteq V_f,$ base of $E = D\}$.

It is readily verified that if $A \subseteq \mathcal{R}^n$ and $B \subseteq \mathcal{R}^m$ are H-cells then their product $A \times B \subseteq \mathcal{R}^{n+m}$ is an H-cell.

Theorem 13. Suppose $A \subseteq \mathcal{R}^n$ and $B \subseteq \mathcal{R}^m$ are H-cells. Suppose $f : A \times B \mapsto \mathcal{R}$ is Riemann integrable. For $x \in A$ let $f_x : B \mapsto \mathcal{R}$ be the function where $f_x(y) = f(x, y)$. Let $U(x) = I_U(f_x)$ and $L(x) = I_L(f_x)$. Then U and L are Riemann integrable, and $\int_{A\times B} f = \int_A L = \int_A U$.

Proof: As usual, it may be supposed that f is nonnegative. Suppose $E^L \subseteq V_f \subseteq E^U$. We may suppose that the H-cells of the base of either E^L or E^U are $\{C_{Ai} \times C_{Bj}\}$ where A (resp. B) is the disjoint union of the C_{Ai} (resp. C_{Bj}). Let h^L_{ij} (resp. h^U_{ij}) be the height over the base $C_{Ai} \times C_{Bj}$, in E^L (resp. E^U). Let E^L_Σ (resp. E^U_Σ) be the element of $\mathcal{E}_b$, where the cells have as bases the C_{Ai}, and for a given such the height is $\sum_j h^L_{ij}$ (resp. $\sum_j h^U_{ij}$). Clearly, $\nu(E^L_f) = \nu(E^L_\Sigma)$ and $\nu(E^U_f) = \nu(E^U_\Sigma)$, and it is easy to see that $E^L_\Sigma \subseteq V_L, V_U \subseteq E^U_\Sigma$, By hypothesis, $\sup\{\nu(E^L_f\} = \inf\{\nu(E^U_f\}$, and the theorem follows. QED.

This theorem is a version of what is known as Fubini's theorem. It is usually stated for closed cells (see theorem 3.10 of [Spivak]); but the theorem for H-cells can be used in various cases, and requires less introduction here of further definitions. In applications, f is often continuous on a closed cell, whence the f_x are all continuous. The roles of A and B may be reversed, so that the integration is performed over A first rather than B in the "double integral". More generally, the integral may be performed one dimension at a time, in any order.

Additional material.

While the Jordan measure is satisfactory for various applications in calculus, the Lebesgue measure has properties of interest in more advanced applications. The Lebesgue measure is an "extension" of the Jordan measure, in that if S is Jordan measurable then it is Lebesgue measurable, and the Lebesgue measure equals the Jordan measure.

The main feature distinguishing the Lebesgue measure from the Jordan measure is the use of infinite collections E of disjoint cells. This results in the complication that the sum $\nu(E) = \sum_i \nu(C_i)$ might be infinite. For this reason, the Lebesgue measure of a measurable set is allowed to be ∞.

Let $\mathcal{E}_\infty$ denote the sets which can be written as the union of a (finite or) countabe collection of disjoint H-cells. For $S \subseteq \mathcal{R}^n$ let $\mu^*(S)$ denote $\inf\{\nu(E) : E \in \mathcal{E}_\infty,\ S \subseteq E\}$. Say that S is Lebesgue measurable if for any set $T \subseteq \mathcal{R}^n$, $\mu^*(T) = \mu^*(S \cap T) + \mu^*(S \cap T^c)$. For a Lebesgue measurable set S, define its Lebesgue measure $\mu_L(S)$ to be $\mu^*(S)$. This definition is technical but convenient; it is due to Caratheodory. A treatment of the Lebesgue measure may be found in any of numerous references, including section 23.2 of [DowdAlg].

The Lebesgue measure has various advantages. It is "countably additive" rather than merely finitely additive. That is, if $\{S_i : i \in \mathcal{N}\}$ is a countable collection of disjoint Lebesgue measurable sets then $\bigcup_i S_i$ is Lebesgue measurable and $\mu_L(\bigcup_i S_i) = \sum_i \mu_L(S_i)$. The "infinite sum" needs to be given a formal definition, which allows ∞ as a value.

For two examples, if $S = \mathcal{Q} \cap [0, 1]$, its Jordan measure is undefined, but its Lebesgue measure is 0. In Riemann integration theory, there is the notion of an "improper integral", which must be defined as a limit. For example, the domain might be an infinite interval. The Lebesgue integral eliminates the need to take a limit.

It is readily seen that S has Lebesgue measure 0 iff for any ϵ there is a cover $E \in \mathcal{E}_\infty$ with $S \subseteq E$ and $\nu(E) < \epsilon$. Such sets are also called null sets. Theorem 9 may be strengthened to the following. Suppose f is a $\mathcal{J}$ function whose domain D is a closed cell; then f is Riemann integrable iff the set of points where f is not continuous is null. See theorem 3.8 of [Spivak].

Exercises.

1. Prove monotonicity and finite subadditivity of a fintely additive measure μ on a ring of sets A.

2. Show that an A-matrix preserves the measure of an H-cell. Hint: First consider the case $n = 2$, S an H-cell with $l_1 = l_2 = 0$, and f the transformation of an A-matrix. $f[S]$ may have a triangle T subtracted from it. Both T and $f[S] - T$ are in $\mathcal{J}$ (subtract a cell from $f[S]$). T may be translated so that the disjoint union equals the original cell.

3. Complete the proof of theorem 7. Given $E \in \mathcal{E}$, first partition the H-cells by the bounding hyperplanes, as in the proof of lemma 1. In the case $V_f \subseteq E$, trim irrelevant cells. In the case $E \subseteq V_f$, fill holes. Finally, in either case, merge cells with the same projection onto the hyperplane $x_{n+1} = 0$.

4. Prove theorem 8. Hint: First prove inequalities involving sets in $\mathcal{E}_b$; for example if $E_1 \subseteq V_f$ and $E_2 \subseteq V_g$, an element $E \in \mathcal{E}_b$ with $E \subseteq V_{f+g}$ and $\nu(E) \geq \nu(E1) + \nu(E_2)$ can be constructed. Second, prove the theorem for $\mathcal{J}^{\geq}$-functions using properties of sup and inf. Third, use arithmetic to prove the theorem for $\mathcal{J}$-functions.

5. Prove lemma 11. Hint: First show that for a step function s, if $I(s) = v$ then $(*)$ holds. Use this to show that for any f, if $\int_a^b f(x)dx = v$ then $(*)$ holds. For the other direction, show that if $(*)$ holds for v then for any $\epsilon > 0$ there are step functions s, t with $I(s) > v - \epsilon$ and $I(t) < v + \epsilon$; choose a subdivision with sufficiently small mesh, and choose sample points where $f(x)$ is sufficiently close to the inf or sup in the subinterval.

9. Complex numbers.

As noted in chapter 1, examples of fields include the rational numbers $\mathcal{Q}$ and the real numbers $\mathcal{R}$. For formal definitions of these systems, and proofs that they are fields, see [DowdBG]. $\mathcal{R}$ is an "extension" of $\mathcal{Q}$ (i.e., contain $\mathcal{Q}$ as a subfield). $\mathcal{R}$ was constructed because $x^2 = 2$ has no solution in $\mathcal{Q}$.

It is readily verified that $x^2 = -1$ has no solution in $\mathcal{R}$ (exercise 1). It is natural to ask whether there is some extension field of $\mathcal{R}$ in which it does. Indeed there is a field which is obtained from the reals by adjoining in the simplest possible way an element i whose square is -1. This field is called the complex numbers, denoted $\mathcal{C}$. It plays a major role in mathematics, including applied mathematics.

The construction of the complex field is quite simple. In $\mathcal{R}^2$, define addition by

$$\langle x_1, x_2\rangle + \langle y_1, y_2\rangle = \langle x_1 + y_1, x_2 + y_2\rangle$$

(i. e. vector addition), and define multiplication by

$$\langle x_1, x_2\rangle\langle y_1, y_2\rangle = \langle x_1y_1 - x_2y_2, x_1y_2 + x_2y_1\rangle.$$

Theorem 1. The complex numbers form a field, denoted $\mathcal{C}$. The elements of the form $\langle x, 0\rangle$ form a copy of the real field $\mathcal{R}$; in particular, $\mathcal{R}$ is a subfield of $\mathcal{C}$.

Proof: We already know from chapter 2 that with vector addition $\mathcal{R}^2$ forms a commutative group, with 0 element $\langle 0, 0\rangle$ and $-\langle x_1, x_2\rangle = \langle -x_1, -x_2\rangle$. The multiplication operation is clearly commutative. Associativity follows using

$$(x_1y_1 - x_2y_2)z_1 - (x_1y_2 + x_2y_1)z_2 = x_1(y_1z_1 - y_2z_2) - x_2(y_1z_2 + y_2z_1)$$

and

$$(x_1y_1 - x_2y_2)z_2 + (x_1y_2 + x_2y_1)z_1 = x_1(y_1z_2 + y_2z_1) + x_2(y_1z_1 - y_2z_2).$$

The multiplicative identity is $\langle 1, 0\rangle$, as is immediately verified. If $\langle x_1, x_2\rangle \neq \langle 0, 0\rangle$ then $x_1^2 + x_2^2 \neq 0$ and

$$\left\langle \frac{x_1}{x_1^2 + x_2^2}, \frac{-x_2}{x_1^2 + x_2^2} \right\rangle$$

is the multiplicative inverse, as is immediately verified. The distributive law follows using

$$x_1(y_1 + z_1) - x_2(y_2 + z_2) = (x_1y_1 - x_2y_2) + (x_1z_1 - x_2z_2)$$

and

$$x_1(y_2 + z_2) + x_2(y_1 + z_1) = (x_1y_2 - x_2y_1) + (x_1z_2 + x_2z_1).$$

The last claim follows because

$$\langle x, 0\rangle + \langle y, 0\rangle = \langle x + y, 0\rangle \quad \text{and} \quad \langle x, 0\rangle\langle y, 0\rangle = \langle xy, 0\rangle;$$

$\langle 0, 0\rangle$ and $\langle 1, 0\rangle$ are of this form; and the additive and multiplicative inverses of complex numbers of this form are also of this form.

Let i denote $\langle 0, 1\rangle$; then $i^2 = \langle -1, 0\rangle$. The complex number $\langle x_1, x_2\rangle$ is equal to $\langle x_1, 0\rangle + \langle x_2, 0\rangle\langle 0, 1\rangle$. Identifying the reals with their copy in $\mathcal{C}$, the complex numbers are exactly the numbers of the form $x_1 + x_2i$ for reals x_1 and x_2. The addition and multiplication laws are seen to be derived using elementary algebra, together with the identity $i^2 = -1$. QED.

$\mathcal{C}$ is a two-dimensional vector space over $\mathcal{R}$ ($\{1, i\}$ comprises a basis). Equipped with the Euclidean norm, it is a normed linear space.

If $x = x_1 + x_2 i$ then its complex conjugate x^* is defined to be $x_1 - x_2 i$. This operation obeys the identities

$(x + y)^* = x^* + y^*$;

$(xy)^* = x^* y^*$;

$x^* = x$ iff x is real;

$x^{**} = x$; and

$xx^* = |x|^2$.

The verification is left as exercise 2.

$\mathcal{C}$ satisfies certain uniqueness properties as an extension of $\mathcal{R}$; see theorem 20.46 of [DowdALG] for example.

Exercises.

1. Show that the equation $x^2 - 1$ has no solution in $\mathcal{R}$. Hint: x^2 is always nonnegative.

2. Prove the properties of the complex conjugation stated in the chapter.

10. Complex differentiation.

A function $f : \mathcal{C} \mapsto \mathcal{C}$ induces a function $\tilde{f} : \mathcal{R}^2 \mapsto \mathcal{R}^2$ (writing a complex number x as $x_1 + ix_2$ where $x_1, x_2 \in \mathcal{R}$, if $y_1 + y_2 i = f(x_1 + x_2 i)$ then $\langle y_1, y_2 \rangle = \tilde{f}(\langle x_1, x_2 \rangle))$. If the function $\tilde{f}$ has a derivative, this is a linear transformation from $\mathcal{R}^2$ to $\mathcal{R}^2$. There is a stronger sense in which f might have a derivative, namely, as a linear transformation from $\mathcal{C}$ to $\mathcal{C}$, where the field of scalars is now considered to be $\mathcal{C}$.

By definition ϕ equals the (complex) derivative of f, iff $f(x + \xi) = f(x) + \phi(\xi) + E$, where $\lim_{\xi \mapsto 0}(E/|\xi|) = 0$. Analogously to the real case, the complex derivative may be considered to be a complex number, and c equals the derivative of f at x iff $f(x + \xi) = f(x) + c\xi + E$, where $\lim_{\xi \mapsto 0}(E/\xi) = 0$, iff $\lim_{\xi \to 0}(f(x + \xi) - f(x))/\xi = c$.

If $f : D \mapsto \mathcal{C}$ where D is an open subset of $\mathcal{C}$, and the complex derivative exists throughout $\mathcal{C}$, $f' : D \mapsto \mathcal{C}$ is the function assigning to x its derivative there (the use of the prime superscript for both the real and complex derivative generally causes no confusion). Differentiation of functions $f : D \mapsto \mathcal{C}^m$ where D is an open subset of $\mathcal{C}^n$ is a topic of interest; but this is omitted here.

The complex derivative has many properties in common with the real derivative. Theorems 4, 5, and 6 of chapter 5 hold, the derivative of a constant function is 0, and the derivative of the identity function is 1, as is easily seen by modifying the proofs as necessary. The derivative of a polynomial or rational function is the same as in the real case, the function now having a complex argument.

Suppose $f : \mathcal{C} \mapsto \mathcal{C}$ is differentiable at x, and $y = f'(x)$. Recall the definition of $\tilde{f}$ from above. From the definition of the derivative, $\lim_{\xi\to 0}(f(x+\xi) - f(x))/\xi = y$ when ξ is real, and it follows that $y_1 = \partial\tilde{f}_1/\partial x$ and $y_2 = \partial\tilde{f}_2/\partial x$. Similarly, $\lim_{\xi\to 0}(f(x+\xi i) - f(x))/(i\xi) = y$ when ξ is real, and it follows that $y_1 = \partial\tilde{f}_2/\partial y$ and $y_2 = -\partial\tilde{f}_1/\partial y$.

Thus, for the complex derivative to exist, the conditions $\partial\tilde{f}_1/\partial x = \partial\tilde{f}_2/\partial y$ and $\partial\tilde{f}_1/\partial y = -\partial\tilde{f}_2/\partial x$ must be satisfied. These are called the Cauchy-Riemann conditions. Exercise 1 gives examples, where they hold, and where they do not.

Exercises.

1. Show that the Cauchy-Riemann conditions hold for $f(x) = x^2$. Show that they do not hold for the "real part" function $f(x_1+ix_2) = x_1$.

11. Power series.

If $\langle x_i : i \in \mathcal{N}\rangle$ is an infinite sequence of complex numbers, define the infinite sequence $\langle s_i\rangle$ by setting $s_i = \sum_{j=0}^{i} x_j$. If the sequence $\langle s_i\rangle$ converges, say to s, we say that the infinite series whose terms are the x_i converges, and call s its sum. The notation $s = \sum_{i=0}^{\infty} x_i$, or just $s = \sum_i x_i$, is used to denote that this is the case. The value s_i is called the i-th partial sum of the series.

An example of a series which converges is the geometric series $1 + x + x^2 + \ldots$ for x with $|x| < 1$. Indeed, $s_i = (1 - x^{i+1})/(1 - x)$, as is easily shown by induction, and this converges to $1/(1 - x)$ if $|x| < 1$. A series is said to diverge if it fails to converge. The following theorem (the comparison test) gives a useful sufficient condition for convergence.

Theorem 1. Suppose $r_i \in \mathcal{R}$ and $r_i \geq 0$ for $i \in \mathcal{N}$, and $\sum_i r_i$ converges to r. Suppose $x_i \in \mathcal{C}$ and $|x_i| \leq r_i$ for $i \in \mathcal{N}$. Then $\sum_i x_i$ converges, and $\sum_i x_i \leq r$.

Proof: For $m \leq n$, $|\sum_{i=m}^{n} x_i| \leq \sum_{i=m}^{n} |x_i| \leq \sum_{i=m}^{n} r_i$. But given ϵ, $\sum_{i=m}^{n} r_i < \epsilon$ for sufficently large m. Thus, the partial sums of $\sum_i x_i$ form a Cauchy sequence. Since $\mathcal{C}$ is a complete metric space, the partial sums converge. Also, $|\sum_{i=0}^{n} x_i| \leq r$ for all n, so $\sum_{i=0}^{\infty} x_i \leq r$. QED.

Two series $\sum_i x_i$ and $\sum_i y_i$ may be added termwise; further if $\sum_i x_i = x$ and $\sum_i y_i = y$ then $\sum(x_i + y_i) = x + y$. This follows since $|\sum_{i=0}^{n}(x_i + y_i) - (x + y)| \leq |\sum_{i=0}^{n} x_i - x| + |\sum_{i=0}^{n} y_i - y|$. Similarly if $\sum_i x_i = x$ then for $c \in C$, $\sum_i cx_i = cx$.

A power series over the scalars $\mathcal{R}$ or $\mathcal{C}$ is a series of the form $\sum_{i\in N} a_i x^i$ where a_i is a scalar. The a_i are called coefficients; the variable x is just a placeholder or notational device, and formally a power series is just the sequence of its coefficients. For each $x \in \mathcal{C}$, the power series becomes a series when x is considered to be the value rather than the placeholder. If this series converges the power series is said to converge at x.

Theorem 2. Given a power series $\sum_i a_i x^i$, let r be the sup of $\{r \in \mathcal{R} : |a_i| \leq 1/r^i$ for sufficiently large $i\}$ if the sup exists, else ∞. If $r = \infty$ then the series converges for all x. Otherwise it converges if $|x| < r$ and diverges if $|x| > r$.

Proof: If $|x| < r$ then $|a_i x^i| < (1-\epsilon)^i$ for some $\epsilon > 0$ and the series converges. If r does not exist then reasoning similarly the series converges for all x. If $|x| > r$ then for infinitely many i $|a_i x^i| > 1$. The series thus diverges, because the partial sums s_i cannot form a Cauchy sequence since $s_i - s_{i-1} = a_i x^i$. QED.

The value r is called the radius of convergence of the series. For example, the radius of convergence of the geometric series $\sum_i x^i$ is 1. The behavior for x with $|x| = r$ depends on the power series. If the series converges for all x, the radius of convergence may be taken as ∞.

Suppose $\sum_i a_i x^i$ is a power series, with radius of convergence $r > 0$ (where r may be ∞); then there is a function f, defined for x with $|x| < r$, by $f(x) = \sum_i a_i x^i$. The notation "let $f(x) = \sum_i a_i x^i$" indicates that f is to be defined in this manner.

Power series may be added or multiplied by a scalar termwise, as follows. $\sum_i a_i x^i + \sum_i b_i x^i = \sum_i (a_i + b_i) x^i$ and $c \sum_i a_i x^i = \sum_i c a_i x^i$. It is readily verified that with these operation the power series form a vector space over the scalars. Also, by the discussion above if $\sum_i a_i x^i = X$ and $\sum_i b_i x^i = Y$ then $\sum_i (a_i + b_i) x^i = X + Y$, and if $\sum_i a_i x^i = X$ then $\sum_i c a_i x^i = cX$.

Define the product of power series $\sum_i a_i x^i$ and $\sum_i b_i x^i$ to be

$$\sum_i \sum_{j+k=i} c_j d_k x^i.$$

Using the identity

$$\sum_{m+l=i} (\sum_{j+k=m} c_j d_k) e_l = \sum_{j+m=i} c_j (\sum_{k+l=m} d_k e_l)$$

it follows that multiplication of power series is associative. Using the identity

$$\sum_{j+k=i} c_j (d_k + e_k) = \sum_{j+k=i} c_j d_k + \sum_{j+k=i} c_j + e_k.$$

it follows that multiplication distributes over addition. Commutativity of multiplication and the existence of a multiplicative identity are readily verified. Thus, the power series comprise a commutative ring with these operations. As already noted, they also form a vector space; this situation is summarized by saying that they form an algebra over the scalars.

Theorem 3. Suppose $|x|$ is less than the radius of convergence of $\sum_i a_i x^i$ and $\sum_i b_i x^i$, and let $X = \sum_i a_i x^i$, $Y = \sum_i b_i x^i$. Then

$$\sum_i (\sum_{j+k=i} c_j d_k) x^i$$

converges, and its sum Z equals XY.

Proof: Let X_i, Y_i, Z_i be the respective partial sums. By properties of convergent sequences X_iY_i converges to XY, so it suffices to show that $E_i = X_iY_i - Z_i$ converges to 0. By hypothesis $|a_i|, |b_i| \leq 1/((1+\epsilon)|x|)^i$ for some $\epsilon > 0$. From this $|E_i| \leq i\sum_{j=i}^{2i} 1/(1+\epsilon)^j$, and this converges to 0. QED.

Theorem 4. Suppose the radius of convergence of $\sum_i a_ix^i$ is r, $f(x) = \sum_i a_ix^i$ for $|x| < r$, and the radius of convergence of $\sum_i(i+1)c_{i+1}x^i$ is R. Then $R = r$, and $\sum_i(i+1)c_{i+1}x^i = f'(x)$ for $|x| < R$.

Proof: By continuity, $\lim_{i\to\infty} c^{1/i} = 1$ for any real c. For $i \geq 1$ let $e_i = i^{1/i} - 1$; by the binomial theorem, $i \geq i(i-1)e_i^2/2$, whence $e_i \leq \sqrt{2/(i-1)}$, and so $\lim_{i\to\infty} i^{1/i} = 1$.

For convenience, for expressions E_i and F_i depending on $i \in \mathcal{N}$, let $E_i \leq_* F_i$ denote the fact that $E_i \leq F_i$ for i sufficiently large.

Suppose $0 < t < s < r$, and $a_i \leq_* 1/s^i$. Then $(i+1)|a_{i+1}| \leq_* (i+1)/s^{i+1} = ((i+1)/s)(1/s^{i+1}) \leq_* (s/t)^i(1/s^i) = 1/t_i$. It follows that $r \leq R$.

Suppose $0 < T < S < R$, and $(i+1)a_{i+1} \leq_* 1/S^i$. Then $|a_{i+1}| \leq_* (1/(i+1))(1/S^i) = (T/(i+1))(1/(TS^i)) \leq_* (S/T)^i(1/(TS^i)) = 1/T^{i+1}$. It follows that $R \leq r$.

Now suppose $|x| < r$, $f(x) = \sum_i a_ix^i$, and $g(x) = \sum_i((i+1)a_{i+1})x^i$. Let f_n, g_n be the partial sums, and let $r_n(x) = f(x) - f_n(x)$; note that $g_n = f_n'$. For y with $|y| < r$, $(f(y) - f(x))/(y-x) - g(x) = T_1 + T_2 + T_3$ where $T_1 = (f_n(y) - f_n(x))/(y-x) - g_n(x)$, $T_2 = (g_n(x) - g(x))$, and $T_3 = (r_n(y) - r_n(x))/(y-x)$. Given $\epsilon > 0$, N_2 can be chosen so that $|T_2| < \epsilon/3$ for $n \geq N_2$. Suppose δ_3 and $s < r$ are such that if $|y-x| < \delta_3$ then $|x|, |y| \leq s$; then if $|y-x| < \delta_3$ then $|T_3| \leq is^{i-1}$, and so there is an $N_3 \geq N_2$ such that if $n \geq N_3$ then $|T_3| \leq \epsilon/3$. Finally, $\delta \leq \delta_3$ can be chosen so that, with $n = N_3$, if $|y-x| < \delta$ then $|T_1| < \epsilon/3$. QED.

If $w_0 \in \mathcal{C}$, a series $\sum_i a_i(w-w_0)^i$ is said to be a power series at w_0. Suppose $f(x) = \sum_i a_ix^i$ for $|x| < r$, and $w_0 \in \mathcal{C}$; then $f(w-w_0) = \sum_i a_i(w-w_0)^i$ for $|w-w_0| < r$. The function $g(w) = f(w-w_0)$ has thus been written as a power series, at the point w_0. The series converges to $g(w)$ for all w such that $|w-w_0| < r$; r is said to be the radius of convergence.

A function $f : D \mapsto \mathcal{C}$ where $D \subseteq \mathcal{C}$ is an open subset is said to be analytic at a point $x_0 \in D$ if there is a power series at x_0, and an $r > 0$, such that $B_{x_0,r} \subseteq D$ and the series converges to $f(x)$ if $|x - x_0| < r$ (note that r must be at most the radius of convergence). f is said to be analytic on D if it is analytic at x_0 for each $x_0 \in D$.

The function $f(x) = 1/x$, for example, is analytic on its domain $D = \mathcal{C} - \{0\}$; the radius of convergence clearly cannot exceed $|x_0|$, and in fact can be made to equal $|x_0|$ (exercise 1).

In $\mathcal{C}$, open balls are often called open discs. A function $f : D \mapsto \mathcal{C}$ where $D \subseteq \mathcal{C}$ is an open subset is said to be holomorphic at a point x_0 if there is an open disc $B_{x_0,r} \subseteq D$ such that $f'(x)$ exists for every $x \in B_{x_0,r}$.

By theorem 4, if $f : D \mapsto \mathcal{C}$ is analytic at x_0 then it is holomorphic at x_0. It is a basic theorem of complex analysis that the converse holds. A proof will be omitted here, as it is not needed. Such can be found in virtually any introductory text in complex analysis; see also [wikiAnHol].

Exercises.

1. Show that if $f(x) = 1/x$, then f has a power series expansion at any $x_0 \neq 0$, with radius of convergence $|x_0|$. Hint: Use the fact that the geometric series $\sum_i x^i$ converges to $1/(1-x)$ if $|x| < 1$, and diverges if $|x| \geq 1$.

12. Transcendental functions.

A function $f : D \mapsto \mathcal{C}$ is said to be algebraic if it is a solution on D of an algebraic equation, i.e., one of the form $\sum_{i=0}^n p_i f^i$ where p_i is a polynomial. Examples of algebraic functions include polynomials, rational functions p/q for polynomials p and q (for which D excludes the roots of q), and $\sqrt{x}$ (common choices for D include $[0,\infty)$ and $\mathcal{C} - (-\infty, 0]$). The derivative of an algebraic function may in many cases be computed using facts from chapter 5. (After suitably formalizing the definition, it can be shown that the derivative of an algebraic function is again algebraic [AnalComb]; the proof in fact gives a method for computing the derivative.)

Functions which are not algebraic are said to be transcendental. Many important functions of mathematics are transcendental. The method of power series may be used to define various such, and determine various of their properties. In this chapter, this will be done for some important transcendental functions; proving that they are transcendental, however, is beyond the scope of the text.

Consider the power series $\sum_i x^i/i!$. It is easily shown that for any real $r > 0$, $1/i! \leq 1/r^i$ for sufficiently large i. Thus, the series converges for all x. Let $\exp(x)$ denote the function defined by the series; this function is called the exponential function.

Theorem 1. $\exp(x+y) = \exp(x)\exp(y)$.

Proof: It follows easily from the binomial theorem that

$$\sum_{j+k=i} \frac{x^j y^j}{j!k!} = \frac{(x+y)^i}{i!}.$$

The theorem follows by formal multiplication of the power series for $\exp(x)$ and $\exp(y)$, and theorem 11.3. QED.

By the completeness of $\mathcal{R}$, if $r \in \mathcal{R}$ then $\exp(r)$ is a real number. Since exp is analytic, it is complex differentiable, hence continuous; and it follows that the function $r \mapsto \exp(r)$ is a continuous function from $\mathcal{R}$ to $\mathcal{R}$. Let e denote the real number $\exp(1)$. By theorem 1 and facts observed in chapter 6, $\exp^q = e^q$ for $q \in \mathcal{Q}$, and so by corollary 5.14, $\exp(r) = e^r$ for $r \in \mathcal{R}$. The constant e assumes considerable importance in the theory of transcendental functions and related areas; it was discovered in the 17th century [wikiConstE]. The value of e to six significant figures is 2.71828.

It has already been shown that e^r is an increasing function whose range is $(0, \infty)$. There is of course a proof directly from the power series: $\exp(0) = 1$; if $0 < r_1 < r_2$ than $\exp(r_1) < \exp(r_2)$ follows because the coefficients of the power series are positive; for $r > 0$ $\exp(r) > 1 + r$; and $\exp(-r) = 1/\exp(r)$.

In this chapter, the notation f' will be used when f is the name of a specific function.

Lemma 2. Suppose $x \in \mathcal{C}$, and $x = x_1 + ix_2$ where $x_1, x_2 \in \mathcal{R}$.

a. $\exp' = \exp$.
b. $\exp(x)^* = \exp(x^*)$.
c. $|\exp(ix_2)| = 1$, and $|\exp(x_1 + ix_2)| = \exp(x_1)$

Proof: Part a follows because the termwise derivative of the power series for exp is the series itself. It is easily seen that $x \mapsto x^*$ is continuous; part b follows because, since the coefficients of the series are real, are real, it holds for the partial sums. For part c, $|\exp(ix_2)|^2 = \exp(ix_2)\exp(-ix_2) = 1$ by part b, and so $|\exp(ix_2)| = 1$; the rest follows by theorem 1 and the fact that $\exp(x_1)$ is a positive real. QED.

Let

$$\cos(x) = \frac{\exp(ix) + \exp(-ix)}{2} = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}$$

and

$$\sin(x) = \frac{\exp(ix) - \exp(-ix)}{2i} = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}.$$

From the definition, the addition law for exp, and the derivative of exp, the following identities are immediate.

$\exp(ix) = \cos(x) + i\sin(x)$
$(\cos(x))^2 + (\sin(x))^2 = 1$
$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
$\sin(x + y) = \cos(x)\sin(y) + \sin(x)\cos(y)$

$\sin' = \cos$ and $\cos' = -\sin$

For real arguments, $\sin : \mathcal{R} \mapsto [0,1]$ and $\cos : \mathcal{R} \mapsto [0,1]$, with $\sin(0) = 0$ and $\cos(0) = 1$. For $\theta \in \mathcal{R}$, $\cos(\theta)$ is the real part, and $\sin(\theta)$ is the imaginary part, of $\exp(i\theta)$.

For the next theorem, it is assumed that the reader is familiar with the mean value theorem and related inequalities. For convenience a treatment is given in the additional material.

Theorem 3. There is a real number π such that $\pi/2$ is the smallest value $\theta \in \mathcal{R}$ such that $\cos(\theta) = 0$. On the domain $[0, \pi/2]$, sin is increasing, cos is decreasing, and sin and cos have range $[0, 1]$.

Proof: Using the facts noted above, and mean value inequalities, it follows successively that

$\cos(\theta) \leq 1$

$\sin(\theta) \leq \theta$

$\cos(\theta) \geq 1 - \theta^2/2$

$\sin(\theta) \geq \theta - \theta^3/6$

$\cos(\theta) \leq 1 - \theta^2/2 + \theta^4/24.$

It follows that $\cos\sqrt{3} < 0$. Since cos is continuous, it follows by corollary 5.17 that there must be a real number p between 0 and $\sqrt{3}$ for which $\cos(p) = 0$. For $\theta \in (0,p)$, $\sin(\theta) \geq \theta(1 - \theta^2/6) \geq \theta/2 > 0$, so cos is decreasing on $[0,p]$ and p is the smallest $\theta > 0$ for which $\cos(\theta) = 0$. Using $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$, and $\sin(\theta) \geq 0$ on $[0,p]$, sin is increasing on $[0,p]$. Using corollary 5.17, sin and cos have range $[0,1]$ on the domain $[0,p]$. Let π be the value $2p$. QED.

Various further transcendental functions may be defined using those defined so far, and various properties proved. This is commonly done in introductory calculus. For convenience a few basic facts will be stated here; any required proofs are left to exercise 1.

The most common transcendental functions may be defined for complex arguments. They are all real valued when restricted to real arguments; from hereon only real arguments are considered. They may be grouped as follows: the exponential function, the six trigonometric functions, and the six hyperbolic trigonometric functions. In table 1, let E denote $\{n\pi : n \in Z\}$, and O $\{(n + 1/2)\pi : n \in Z\}$.

Taking the inverse f^{-1} of each of these functions f yields further common transcendental functions ($\exp^{-1}$ is custumarily written as ln). The range must be specified, since in general, given y, there are multiple arguments x such that $f(x) = y$. The range choice for f given below may be one of various possibilities; sometimes another choice is better for a particular use. In table 2, if f' is marked $*$, ± 1 are removed from its domain.

f	domain	range	definition	f'
$\exp(x)$	$\mathcal{R}$	$(0,\infty)$	power series	$\exp(x)$
$\sin(x)$	$\mathcal{R}$	$[-1,1]$	power series	$\cos(x)$
$\cos(x)$	$\mathcal{R}$	$[-1,1]$	power series	$-\sin(x)$
$\tan(x)$	$\mathcal{R}-O$	$\mathcal{R}$	$\frac{\sin(x)}{\cos(x)}$	$(\sec(x))^2$
$\cot(x)$	$\mathcal{R}-E$	$\mathcal{R}$	$\frac{\cos(x)}{\sin(x)}$	$-(\csc(x))^2$
$\sec(x)$	$\mathcal{R}-O$	$\mathcal{R}-(-1,1)$	$\frac{1}{\cos(x)}$	$\tan(x)\ \sec(x)$
$\csc(x)$	$\mathcal{R}-E$	$\mathcal{R}-(-1,1)$	$\frac{1}{\sin(x)}$	$-\cot(x)\ \csc(x)$
$\sinh(x)$	$\mathcal{R}$	$\mathcal{R}$	$\frac{e^x-e^{-x}}{2}$	$\cosh(x)$
$\cosh(x)$	$\mathcal{R}$	$[1,\infty)$	$\frac{e^x+e^{-x}}{2}$	$\sinh(x)$
$\tanh(x)$	$\mathcal{R}$	$(-1,1)$	$\frac{\sinh(x)}{\cosh(x)}$	$(\text{sech}(x))^2$
$\coth(x)$	$\mathcal{R}-\{0\}$	$\mathcal{R}-[-1,1]$	$\frac{\cosh(x)}{\sinh(x)}$	$-(\text{csch}(x))^2$
$\text{sech}(x)$	$\mathcal{R}$	$(0,1]$	$\frac{1}{\cosh(x)}$	$-\tanh(x)\ \text{sech}(x)$
$\text{csch}(x)$	$\mathcal{R}-\{0\}$	$\mathcal{R}-\{0\}$	$\frac{1}{\sinh(x)}$	$-\coth(x)\ \text{csch}(x)$

Table 1

f	domain	range	f'
$\ln(y)$	$(0,\infty)$	$\mathcal{R}$	$\frac{1}{y}$
$\sin^{-1}(y)$	$[-1,1]$	$[-\pi/2,\pi/2]$	$\frac{1}{\sqrt{1-y^2}}(*)$
$\cos^{-1}(y)$	$[-1,1]$	$[0,\pi]$	$-\frac{1}{\sqrt{1-y^2}}(*)$
$\tan^{-1}(y)$	$\mathcal{R}$	$(-\pi/2,\pi/2)$	$\frac{1}{1+y^2}$
$\cot^{-1}(y)$	$\mathcal{R}$	$(0,\pi)$	$-\frac{1}{1+y^2}$
$\sec^{-1}(y)$	$\mathcal{R}-(-1,1)$	$[-\pi/2,\pi/2]$	$\frac{1}{y\sqrt{y^2-1}}(*)$
$\csc^{-1}(y)$	$\mathcal{R}-(-1,1)$	$[0,\pi]$	$-\frac{1}{y\sqrt{y^2-1}}(*)$
$\sinh^{-1}(y)$	$\mathcal{R}$	$\mathcal{R}$	$\frac{1}{\sqrt{y^2+1}}$
$\cosh^{-1}(y)$	$[1,\infty)$	$[0,\infty)$	$\frac{1}{\sqrt{y^2-1}}(*)$
$\tanh^{-1}(y)$	$(-1,1)$	$\mathcal{R}$	$\frac{1}{1-y^2}$
$\coth^{-1}(y)$	$\mathcal{R}-[-1,1]$	$\mathcal{R}-\{0\}$	$-\frac{1}{y^2-1}$
$\text{sech}^{-1}(y)$	$(0,1]$	$[0,\infty)$	$-\frac{1}{y\sqrt{1-y^2}}(*)$
$\text{csch}^{-1}(y)$	$\mathcal{R}-\{0\}$	$R-\{0\}$	$-\frac{1}{y\sqrt{y^2+1}}$

Table 2

Among other uses, these functions provide antiderivatives for algebraic functions, which do not have an algebraic antiderivative, for example $1/\sqrt{1-x^2}$. The process of obtaining an anti-derivative is known as "symbolic integration" [wikiSymbInt]. Many may be obtained using tables (see [wikiIntLists]). There is also free software for obtaining antiderivatives, Axiom (see [wikiAxiom]) for example. It is still valuable to students to master traditional methods, such as substitution and integration by parts, for obtaining antiderivatives "by hand". The example $\sqrt{1-x^2}$ will be given; as will shortly be seen, this antiderivative is a very useful one.

Recall the notation from introductory calculus, that $\int f(x)dx$ denotes an antiderivative of f. Letting $\phi = 2\theta$,
$\int \cos(2\theta)\, d\theta = (1/2)\int \cos\phi\, d\phi = (1/2)\sin(\phi) = \sin(\theta)\cos(\theta)$.
Letting $x = \cos(\theta)$, $\int \sqrt{1-x^2}\, dx = \int \cos(\theta)\, d\sin(\theta) = \int (\cos(\theta))^2\, d\theta = \int (1/2)(1+2\cos(\theta))\, d\theta = (1/2)(\theta + \sin(\theta)\cos(\theta) = (1/2)(\sin^{-1}(x) + x\sqrt{1-x^2})$.

Theorem 4. The area (Jordan measure) of the circular disc $S = \{x \in \mathcal{R}^2 : |x - x_0| \leq r\}$ equals πr^2.

Proof: By theorem 8.5, it may be assumed that $x_0 = 0$. Let $S_\geq = \{x \in S : x_2 \geq 0\}$, $S_> = \{x \in S : x_2 > 0\}$, $S_< = \{x \in S : x_2 < 0\}$, and $S_0 = \{x \in S : x_2 = 0\}$. It is readily verified that $\mu(S_0) = 0$, so $\mu(S_\geq) = \mu(S_>)$. By theorem 5.6 $\mu(S_<) = \mu(S_>)$. Thus, $\mu(S) = 2\mu(S_\geq)$. By theorem 8.7 $\mu(S_\geq) = \int_{[-r,r]} \sqrt{r^2 - x^2}$. An antiderivative for $\sqrt{r^2-x^2}$ is $(r^2/2)(\sin^{-1}(x/r) + (x/r)\sqrt{1-(x/r)^2})$. By theorem 8.12, $\mu(S_\geq) = (1/2)\pi r^2$. QED.

Theorem 5. The volume (Jordan measure) of the spherical ball $S = \{x \in \mathcal{R}^3 : |x - x_0| \leq r\}$ equals $(4/3)\pi r^2$.

Proof: By arguments as in theorem 4,

$$\mu(S) = 2\int_{[-r,r]\times[-r,r]} f(x,y)$$

where $f(x,y) = \sqrt{r^2 - x^2 - y^2}$ if $x^2 + y^2 \leq r^2$, else 0. By theorem 8.13,

$$\mu(S) = 2\int_{[-r,r]} \int_{[-\sqrt{r^2-x^2},\sqrt{r^2-x^2}]} \sqrt{r^2 - x^2 - y^2}.$$

As seen in the proof of theorem 4, the inner integral equals $(\pi/2)(r^2 - x^2)$. $r^2x - x^3/3$ is an antiderivative for $r^2 - x^2$, and the outer integral equals $(2/3)\pi r^3$. QED.

Additional material.

Theorem A1. Suppose $[l,u]$ is a closed interval in $\mathcal{R}$, $f : [l,u] \mapsto \mathcal{R}$ is continuous, and $f'(t)$ exists for $t \in (l,u)$.

a. If $L \leq f'(t) \leq M$ for $t \in (l,u)$ then $L(u-l) \leq p(u) - p(l) \leq M(u-l)$.
b. If $f'(t) \geq 0$ (resp. $f'(t) \leq 0$) for $t \in (l,u)$ then f is nondecreasing (resp. nonincreasing).

c. If $f'(t) > 0$ (resp. $f'(t) < 0$) for $t \in (l, u)$ then f is increasing (resp. decreasing).

Proof: For part a, the proof that $p(u)-p(l) \leq M(u-l)$ is a modified version of the proof of lemma 4.10 (simply replace $|f(u)-f(l)|$ by $f(u)-f(l)$). To prove that $L(u-l) \leq p(u)-p(l)$, note that $(-f)'(t) \leq (-L)$, whence $(-f)(u)-(-f)(l) \leq (-L)(u-l)$. For part b, If $f'(t) \geq 0$ then $f(u)-f(l) \geq 0$ by part a; and $[l, u]$ can be replaced by any $[l', u'] \subseteq [l, u]$. The case $f'(t) \leq 0$ follows similarly. For part c when $f'(t) > 0$, suppose $l \leq a < b \leq u$ are such that $f(a) = f(b)$; f is nondecreasing by part b, and it follows that $f(x) = f(a)$ for $x \in [a, b]$, and so $f'(x) = 0$ for $x \in (a, b)$, a contradiction. The case $f'(t) < 0$ follows similarly. QED.

This theorem is all that is required for theorem 3. The mean value theorem states that $(f(u)-f(l))/(u-l) = f'(t)$ for some $t \in [l, u]$. Various authors have suggested avoiding the use of the mean value theorem as much as possible in favor of theorem A1. For convenience, a proof of the mean value theorem will be given. Note that if f' is continuous, it follows by corollary 5.17; but the additional hypothesis is not necessary.

Lemma A2. Suppose f is defined and differentiable on an open interval I, $x \in I$, and either (1) $f(x) \geq f(y)$ for $y \in I$ or (2) $f(x) \leq f(y)$ for $y \in I$. Then $f'(x) = 0$.

Proof: In case 1, for y sufficiently close to x, $(f(y)-f(x))/(y-x)$ is ≤ 0 for $y > x$, and ≥ 0 for $y < x$. The derivative must thus be 0. A similar argument holds in case 2. QED.

Theorem A3. Suppose $[l, u]$ is a closed interval in $\mathcal{R}$, $f : [l, u] \mapsto \mathcal{R}$ is continuous, and $f'(t)$ exists for $t \in (l, u)$. Letting m denote $(f(u)-f(l))/(u-l)$, $f'(t) = m$ for some $t \in [l, u]$.

Proof: Let $h(t) = f(l) + m(t-l)$; replacing f by $f + h$ it suffices to prove the theorem when $f(u) = f(l)$ and $m = 0$. By theorem 5.8.b there is an t_1 such that $f(t_1)$ is the largest value attained by f on $[l, u]$. If $t_1 \neq l, u$ then $f'(t_1) = 0$ by lemma A2. Otherwise let t_2 be such that f attains its smallest value on $[l, u]$ at t_2. If $t_2 \neq l, u$ then $f'(t_2) = 0$. In the remaining case f is constant on $[l, u]$ and $f'(t) = 0$ for any $t \in [l, u]$. QED.

Exercises.

1. Show that the for the definitions of the common transcendental functions given above, each function is defined and differentiable on its domain, has the specified range, and that the derivatives are as specified. Hint: The derivative of an inverse function can be computed using the chain rule. Since $f(f^{-1}(y)) = y$, $f'(f^{-1}(y))(f^{-1})'(y) = 1$; writing x for $f^{-1}(y)$ this may also be written $dx/dy = 1/(dy/dx)$. For example,

$(d/dy)(\ln(y)) = 1/\exp(x) = 1/y$, and $(d/dy)(\sin^{-1}(y)) = 1/\cos(x) = 1/\sqrt{1-y^2}$.

13. Multivariable substitution.

In this chapter, the theorem on substitution in multivariable integration will be proved. The proof requires a fair amount of additional material. According to [wikiIntSub], the theorem was only proved in the 1890's. Readers can skip the proofs and read them later. Various facts given in the proof are of wide utility.

Recall from chapter 5 the "strong norm" $|f|_s = \sup\{|f(x)|: |x| = 1\}$, of a linear function $f \in L(\mathcal{R}^n; \mathcal{R}^m)$.

Lemma 1.

a. $|f|_s$ is a norm on $L(\mathcal{R}^n; \mathcal{R}^m)$ (in particular the sup is finite).
b. $|f|_s$ is equivalent to the Euclidean norm.
c. If $f \in L(\mathcal{R}^n; \mathcal{R}^m)$ and $g \in L(\mathcal{R}^m; \mathcal{R}^l)$ then $|g \circ f|_s \le |g|_s|f|_s$.

Proof: Exercise 2. QED.

For this chapter $L(\mathcal{R}^n; \mathcal{R}^n)$ will be assumed to be equipped with the strong norm. Recall from chapter 7 that the invertible linear transformations form a group $GL_n \subseteq L(\mathcal{R}^n; \mathcal{R}^n)$. In the usual manner, GL_n is a metric subspace (but not a vector subspace!).

Lemma 2.

a. If $\phi \in GL_n$, $\psi \in L(\mathcal{R}^n; \mathcal{R}^n)$, and $|\phi - \psi|_s < 1/|\phi^{-1}|_s$ then $\psi \in GL_n$; in particular GL_n is an open subspace of $L(\mathcal{R}^n; \mathcal{R}^n)$.
b. The map $\phi \mapsto \phi^{-1}$ is a homeomorphism from GL_n to itself.

Proof: $|x| = |\phi^{-1}(\phi(x))| \le |\phi^{-1}|_s|\phi(x)|$; thus, $|\phi(x)| \ge |x|/|\phi^{-1}|_s$. So $|\psi(x)| \ge |\phi(x)| - |(\phi - \psi)(x)| \ge (1/|\phi^{-1}|_s - |\phi - \psi|_s)|x|$. It follows that if $|\phi - \psi|_s < 1/|\phi^{-1}|_s$ then ψ is injective, whence $\psi \in GL_n$.

For part b, writing α for $1/|\phi^{-1}|_s$ and β for $|\phi - \psi|_s$, as just shown $|\psi(x)| \ge (\alpha - \beta)|x|$. Replacing x by $\psi^{-1}(y)$, $|y| \ge (\alpha - \beta)|\psi^{-1}(y)|$, whence $|\psi^{-1}(y)|_s \le 1/(\alpha - \beta)$. Then $|\psi^{-1} - \phi^{-1}|_s = |\psi^{-1}(\phi - \psi)\phi^{-1}|_s \le |\psi^{-1}|_s|(\phi - \psi|_s|\phi^{-1}|_s \le \beta/(\alpha(\alpha - \beta))$. So $|\psi^{-1} - \phi^{-1}|_s$ approaches 0 as $\beta = |\phi - \psi|_s$ does, and $\phi \mapsto \phi^{-1}$ is continuous. Since ϕ is its own inverse, ϕ is a homeomorphism. QED.

Suppose $f : D \mapsto \mathcal{R}$ where $D \subseteq \mathcal{R}$ is an open subset. Suppose $x \in D$, $f'(x)$ exists, and its value s is nonzero. Let $E(h) = f(x + h) - f(x) - sh$, and suppose δ is such that if $|h| < \delta$ then $|E(h)/h| < |s|$. Then $f(x+h)-f(x) = (s+E(h)/h)h$ is nonzero, that is, $f(x+h) \ne f(x)$ for $|h| < \delta$. Refining this argument yields the following, known as the inverse function theorem.

Theorem 3. Suppose $D \subseteq \mathcal{R}^n$ is an open subset, $f : D \mapsto \mathcal{R}^n$ is a C_1 function, $x_0 \in D$, and $f'(x_0)$ is invertible. Then there is an open subset $U \subseteq D$ such that $x_0 \in U$ and the following hold.

a. $f \restriction U$ is injective.
b. $V = f[U]$ is an open subset of $\mathcal{R}^n$.
c. The inverse function $g : V \mapsto U$ is C_1.

Proof: Let ϕ denote $f'(x_0)$ and let ν denote $1/|\phi^{-1}|_s$. Using the hypothesis that f is C_1, choose δ such that if $x \in B_{x_0\delta}$ then $|f'(x) - f'(x_0)|_s < \nu/2$. Suppose $x_1, x_1 + \xi \in B_{x_0\delta}$, and let $p(t) = f(x_1 + t\xi) - tf'(x_0)(\xi)$. Then $|p'(t)| = |f'(x_1 + t\xi)(\xi) - f'(x_0)(\xi)| \leq (\nu/2)|\xi| = (\nu/2)|\phi^{-1}(\phi(\xi)) \leq (\nu/2)|\phi^{-1}|_s|\phi(\xi)| = (1/2)|\phi(\xi)|$. By lemma 4.10, $|p(1)-p(0)| \leq (1/2)|\phi(\xi)|$, that is, $|f(x_1+\xi)-\phi(\xi)-f(x_1)| \leq (1/2)|\phi(\xi)|$, whence $|f(x_1 + \xi) - f(x_1)| \geq (1/2)|\phi(\xi)|$. But $|\phi|_s \geq \nu$, and so $|f(x_1 + \xi) - f(x_1)| \geq (\nu/2)|\xi|$, proving part a, with $U = B_{x_0\delta}$.

Suppose $x_1 \in B_{x_0\delta}$ and choose ρ with $0 < \rho < \delta - |x_1 - x_0|$, so that $B^{\mathrm{cl}}_{x_1\rho} \subseteq B_{x_0\delta}$. Let $y_1 = f(x_1)$ and suppose $y \in B_{y_1,\nu\rho/4}$. For $x \in B^{\mathrm{cl}}_{x_1\rho}$ let $e(x) = |y-f(x)|$. Since $B^{\mathrm{cl}}_{x_1\rho}$ is compact and e is continuous (exercise 1), there is a value $x_m \in B^{\mathrm{cl}}_{x_1\rho}$ such that $e(x_m) \leq e(x)$ for all $x \in B^{\mathrm{cl}}_{x_1\rho}$. Now, $|f(x) - f(x_1)| \leq e(x) + e(x_1) < e(x) + \nu\rho/4$; and by an argument of part a if $|x - x_1| = \rho$ then $|f(x) - f(x_1)| \geq \nu\rho/2$, whence $e(x) + \nu\rho/4 > \nu\rho/2$, whence $e(x) > \nu\rho/4$. But $e(x_1) < \nu\rho/4$, and it follows that $x_m \in B_{x_1\rho}$.

Let $\zeta = y - f(x_m)$, and let $\xi = \phi^{-1}(\zeta)$. Choose $t \in (0,1)$ with $x_m + t\xi \in B_{x_1\rho}$. From the definitions, $|f(x_m) - y + \phi(t\xi)| = (1-t)\zeta$. By an argument of part a, $|f(x_m + t\xi) - f(x_m) - \phi(t\xi)| \leq (1/2)t|\zeta|$. It follows that $e(x_m+t\xi) = |f(x_m+t\xi)-y| \leq (1-t/2)|\zeta| = (1-t/2)e(x_m)$. If $e(x_m) \neq 0$ then $e(x_m + t\xi) < e(x_m)$, a contrdiction. Thus, $e(x_m) = 0$, so $f(x_m) = y$. Since y was arbitrary, $B_{y_1,\nu\rho/4} \subseteq f[B_{x_1\rho}]$. Since x_1 was arbitrary, $f[B_{x_0\delta}]$ is open. This proves part b.

For part c, first note that if $x \in U = B_{x_0\delta}$ then $|f'(x) - \phi|_s < 1/(2|\phi|_s)$, so by lemma 2.a $f'(x)$ is invertible; let ψ denote $f'(x)$. Suppose $y, y + \eta \in V$, $x = g(y)$, and $\xi = g(y + \eta) - x$. Then $\zeta = f(x + \xi) - f(x) = \psi(\xi) + E(\xi)$ where $E(\xi)$ goes to 0 faster than ξ; thus $\psi^{-1}(\zeta) + \psi^{-1}(E(\xi))$, i.e., $g(y + \zeta) - g(y) = \psi^{-1}(\zeta) - \psi^{-1}(E(\xi))$. By an argument of part a, $\zeta \geq (\nu/2)|\xi|$, whence $|\psi^{-1}(E(\xi))|/|\zeta| \leq (|\psi^{-1}|_s/(\nu/2))/(E(\xi)|/|\xi|)$, whence $|\psi^{-1}(E(\xi))|$ goes to 0 faster than ζ, whence $g'(y)$ exists, and in fact equals $f'(x)^{-1} = f'(g(y))^{-1}$. g is continuous since it is differentiable, f' is continuous by hypothesis, and $\psi \mapsto \psi^{-1}$ is continuous by lemma 2.b. Thus, g' is continuous. QED.

If U and V are open subsets of $\mathcal{R}^n$, a function $f : U \mapsto V$ is said to be a diffeomorphism if f is bijective and both f and f^{-1} are differentiable. In particular it is a homeomorphism. A diffeomorphism is said to be C_1 if both f and f^{-1} are.

Corollary 4. Suppose $f : D \mapsto \mathcal{R}^n$ where $D \subseteq \mathcal{R}^n$ is open, and f is C_1, injective, and nonsingular. Then $f[D]$ is an open subset of $\mathcal{R}^n$, and

f is a C_1-diffeomorphism to $f[D]$.

Proof: f is a bijection to $f[D]$; let f^{-1} be the inverse function. The function g in the statement of theorem 3 must equal $f^{-1} \restriction V$. For any $y_0 \in f[D]$ let $x_0 = f^{-1}(y_0)$. Then $y_0 \in V$ and V is open, which shows that $f[D]$ is open. By theorem 3.c f^{-1} is C_1 at y_0. QED.

Lemma 5. A diffeomorphism is nonsingular.

Proof: Suppose $g \circ f = \iota$. Then for any $x \in D$, $g'(f(x))f'(x)) = \iota$, so $f'(x)$ is nonsingular. QED.

In particular, if f satisfies the hypotheses of corollary 4 then f^{-1} is nonsingular, indeed $(f^{-1})'(y) = f'(f^{-1}(y))^{-1}$. Some of the conclusions of corollary 4 can be drawn from weaker hypotheses. The theorem of "invariance of domain" states that if $D \subseteq \mathcal{R}^n$ is open, and $f : D \mapsto \mathcal{R}^n$ is continuous and injective, then $f[D]$ is an open subset, and f is a homeomorphism to $f[D]$. The proof of this theorem is too lengthy for an introductory text.

Lemma 6. Suppose $q : X \mapsto Y$ is a homeomorphism between topological spaces, and $S \subseteq X$. Then $q[S]^{\text{int}} = q[S^{\text{int}}]$, $q[S]^{\text{cl}} = q[S^{\text{cl}}]$, and $q[S]^{\text{bd}} = q[S^{\text{bd}}]$.

Proof: This follows because q and q^{-1} preserve $\subseteq$, etc.; and open sets, etc. Details are left to exercise 3. QED.

Lemma 7. If $S \subseteq \mathcal{R}^n$ bounded then S is Jordan measurable iff S^{bd} has Jordan measure 0.

Proof: For S Jordan measurable it has already been seen that S^{bd} has Jordan measure 0, in lemma 8.4. Conversely, Given ϵ, let $E^B \in \mathcal{E}$ be such that $S^{\text{bd}} \subseteq E^B$ and $\nu(E^B) < \epsilon$. Subdivide a sufficently large H-cell by the cutting hyperplanes of E^B. Let E^I be the union of the H-cells which intersect S, but are not in E^B. Let $E^O = E^I \cup E^B$. The lemma follows using lemma 8.2. QED.

Lemma 8. Suppose $U \subseteq \mathcal{R}^n$ is open and $D \subseteq U$ is compact. Then there is an open set W such that $D \subseteq W$, W^{cl} is compact, and $W^{\text{cl}} \subseteq U$.

Proof: For $x \in D$ choose δ such that $B^{\text{cl}}_{x\delta} \subseteq U$. Since D is compact it is covered by finitely many of the $B_{x\delta}$; let W be the union of these. QED.

Lemma 9. Suppose $U \subseteq \mathcal{R}^n$ is open and $q : U \mapsto \mathcal{R}^n$ is a C_1-diffeomorphism to $q[U]$. Suppose $D \subseteq U$ is Jordan measurable. Then $q[D]$ is Jordan measurable.

Proof: By lemmas 6 and 7, it suffices to show that $\mu(q[D]^{\text{bd}}) = \mu(q[D^{\text{bd}}]) = 0$, given that $\mu(D^{\text{bd}}) = 0$. Let W be as in lemma 8; then $|q'(x)|_s$ is bounded on W^{cl}, say by b. Using lemma 4.10, it follows that for $x, y \in W$, $|q(y) - q(x)| \leq b|y - x|$ (exercise 4). Given ϵ, let

$\delta = \epsilon/(b\sqrt{n})^n$. Since $\mu(D^{\rm bd}) = 0$ there is an $E \in \mathcal{E}$ with $D^{\rm bd} \subseteq E$ and $\mu(E) < \delta$. In fact, E may be chosen to be a disjoint union of N "cubical" cells each of width w, i.e., where $u_i - l_i = w$ for $1 \le i \le n$ (exercise 5). If C is one of the cells, then $\nu(C) = w^n$, and $Nw^n < \delta$. Also, $\mu(q[C]) \le (b\sqrt{n}w)^n$, because C is contained in $B_{c,\sqrt{n}(w/2)}$ where c is the center of C, $q[B_{xr}] \subseteq B_{q(x),br}$, and $B_{ys} \subseteq C'$ where C' is the cubical cell with center y and width $2s$. It follows by subadditivity that $\mu(q[D^{\rm bd}]) \le N(b\sqrt{n}w)^n$, so $\mu(q[D^{\rm bd}]) < (b\sqrt{n})^n\delta = \epsilon$. QED.

The bound on $\mu(f[C])$ in the preceding proof could be decreased, by computing the Jordan measure of a ball. This value is useful in some subjects; see the additional material for further remarks.

The notion of integration by change of variable, or substitution, should be familiar to students after first year calculus. The following theorem provides the theoretical justification in the case where the domain of integration is an interval.

Lemma 10. Suppose $U \subseteq \mathcal{R}$ is open, $p : U \mapsto \mathcal{R}$ is C_1, $[a, b] \subseteq U$, and $f : p[[a, b]] \mapsto \mathcal{R}$ is continuous. Then $\int_a^b f(p(u))p'(u)du$ equals $\int_{p(a)}^{p(b)} f(x)dx$ if $p(a) \le p(b)$, or $-\int_{p(b)}^{p(a)} f(x)dx$ if $p(a) > p(b)$.

Proof: Suppose $p[[a, b]] = [c, d]$. Choose $c_1 < c$ and $d_1 > d$. Let $f_1(x) = f(c)$ if $c_1 \le x < c$, $f_1(x) = f(x)$ if $c \le x \le d$, and $f_1(x) = f(d)$ if $d < x \le d_1$. For $x \in [c_1, d_1]$ let $F(x) = \int_{c_1}^x f(x)dx$. If $p(a) \le p(b)$ then $\int_{p(a)}^{p(b)} f(x)dx = F(p(b)) - F(p(a))$, and if $p(a) > p(b)$ then $\int_{p(b)}^{p(a)} f(x)dx = -(F(p(b)) - F(p(a))$. By theorem 4.7, for $u \in [a, b]$, $(F \circ p)'(u) = f(p(u)) \cdot p'(u)$. By theorem 8.12, $\int_a^b f(p(u))p'(u)du = F(p(b)) - F(p(a))$. QED.

Corollary 11. Suppose $U \subseteq \mathcal{R}$ is open, $p : U \mapsto \mathcal{R}$ is C_1 and injective, $[a, b] \subseteq U$, and $f : p[[a, b]] \mapsto \mathcal{R}$ is continuous. Then $\int_{[a,b]} f(p'(u))$ $|p'(u)|du$ equals $\int_{p[[a,b]]} f(x)dx$.

Proof: By corollary 6.18, p is either increasing or decreasing. Now, if a function h is differentiable at x and increasing (resp. decreasing) in some open interval containing x then $h'(x) \ge 0$ (resp. $h'(x) \le 0$), directly from the definition of the derivative. It follows that $|p'(x)| = p'(x)$ if p is increasing, and $|p'(x)| = -p'(x)$ if p is decreasing. QED.

Lemma 12. Suppose $U \subseteq \mathcal{R}$ is open, $p : U \mapsto \mathcal{R}$ is C_1 and injective, $D \subseteq U$ is Jordan measurable and compact, and $f : p[D] \mapsto \mathcal{R}$ is continuous. Then $\int_D (f \circ p') \cdot |p'|$ equals $\int_{p[D]} f$.

Proof: By lemma 9 $p[D]$ is Jordan measurable, and by theorem 5.7 f is bounded, so by corollary 8.10 f is Riemann integrable on $p[D]$. Choose $E \in \mathcal{E}$ with $E \subseteq p[D]$. Let s be a "step function" on E with

$s \leq f$. Let $\{C\}$ be the H-cells of E, or indifferently their closures; let s_C be the value of s on C. By corollary 11, $s_C \cdot \nu(C) = \int_{p^{-1}[C]}(s_C \circ p) \cdot |p'|$. So $\sum_C s_C \cdot \nu(C) = \sum_C \int_{p^{-1}[C]}(s_C \circ p) \cdot |p'| \leq \sum_C \int_{p^{-1}[C]}(f \circ p) \cdot |p'| = \int_{p^{-1}[E]}(f \circ p) \cdot |p'| \leq \int_D (f \circ p) \cdot |p'|$. It follows that $\int_{p[D]} f \leq \int_D (f \circ p') \cdot |p'|$.

Choosing $E \in \mathcal{E}$ with $p[D] \subseteq E \subseteq p[U]$, and $s \geq f$, and arguing similarly yields $\int_{p[D]} f \geq \int_D (f \circ p') \cdot |p'|$. To see that E can be chosen, for $x \in D$ there is an open interval U_x with $x \in U_x$ and $U_x^{\mathrm{cl}} \subseteq p[U]$. Let E be the union of the H-cells corresponding to the open intervals of a finite subcover of $\{U_x\}$. QED.

If $f : D \mapsto \mathcal{R}^n$ where $D \subseteq \mathcal{R}^n$ is open, and f is differentiable at $x \in D$, the notation $f'(x)$ may be used indifferently to denote the matrix; in particular the determinant $\det(f'(x))$ may be taken. This real value is often called the Jacobian of f at x. From theorem 8.6, it represents up to sign the "volume scale factor" of the transformation f at the point x. The notation J^f will be used for the function $x \mapsto \det(f'(x))$.

Generalizing a notation of chapter 8, $\int_D f(x)\, dx$ will be used as an alternative notation for $\int_D f$; the advantage is that f may be a term involving several functions, and a separate name need not be defined.

Theorem 13. Suppose $p : U \mapsto \mathcal{R}^n$ is a C_1-diffeomorphism from an open subset $U \subseteq \mathcal{R}^n$ to $f[U]$, $D \subseteq p[U]$ is Jordan measurable and compact, and $f : D \mapsto \mathcal{R}$ is continuous. Then $\int_D f = \int_{p^{-1}[D]} f(p(x))|J^p(x)|\, dx$.

Proof: The proof is by induction on n. The basis $n = 1$ follows by lemma 12. Suppose $n > 1$ and the theorem holds for $n - 1$. The proof will be given as a series of numbered facts. Let $T(p, D, f)$ denote the statement that the theorem holds for p, D, and f.

1. If $T(p_1, D, f)$ and $T(p_2, p_1^{-1}[D], f \circ p_1)$ then $T(p_1 \circ p_2, D, f)$. Indeed, $\int_D f = \int_{p_1^{-1}[D]} f(p_1(x)|J^{p_1}(x)|\, dx = \int_{p_2^{-1}[p_1^{-1}[D]]} f(p_1(p_2(x)))$ $|J^{p_1}(p_2(x))|\, |J^{p_2}(x)|\, dx = \int_{p^{-1}[D]} f(p(x))|J^p(x)|\, dx$. Remaining details are left to exercise 6.

2. $T(p, D, f)$ if p is a linear transformation. Let $\tilde{p}$ be p in the first n components, and 1 in the $(n+1)$ component. Using theorems 8.6 and 8.7, $\int_{p^{-1}[D]} f(p(x))|J^p(x)|\, dx = |\det(p)| \int_{p^{-1}[D]} f \circ p = |\det(p)|\mu(V_{f \circ p}) = |\det(p)|\, |\det(\tilde{p}^{-1})|\mu(V_f) = \mu(V_f)$.

3. Suppose $u \in U$. Since $J^p(u) \neq 0$, $p_n^{\partial_i}(u) \neq 0$ for some i. Recall the notation $P^{(in)}$ for the permutation matrix for the transposition (in) (let $P^{(in)}$ denote the identity matrix if $i = n$). Let $\hat{p} = p \circ P^{(in)}$. Using the inverse function theorem, an open cell C with $u \in C$ can be found such that $C^{\mathrm{cl}} \subseteq U$, and, letting $q(x) = \langle x_1, \ldots, x_{n-1}, \hat{p}_n(x) \rangle$, there is an open subset U_C with $C^{\mathrm{cl}} \subseteq U_C \subseteq U$ such that $q \restriction P^{(in)}[U_C]$

is a C_1-diffeomorphism. Writing $q(x)$ for y, $r(y) = r(q(x)) = \hat{p}(x)$, so $r_n(y) = \hat{p}_n(x) = q_n(x) = y_n$. For future reference, let $B = q[\hat{C}]$, $A = r[B] = \hat{p}[\hat{C}] = p[C]$. Let $\hat{C} = J \times [l_C, u_C]$ where J is an open cell in $\mathcal{R}^{n-1}$, and $B = J \times [l_B, u_B]$. Let f_A be the function with domain A such that $f_A(x) = f(x)$ if $x \in D$, else 0.

4. In the notation of fact 3, $T(r, A, f_A)$. To see this, let $\check{K}$ be an interval in $\mathcal{R}^{n-1}$ such that $A \subseteq K = \check{K} \times [l_B, u_B]$. Let f_K be the function with domain K such that $f_K(x) = f(x)$ if $x \in D$, else 0. Let $K_{x_n} = \{\langle x_1, \ldots, x_{n-1}\rangle : \langle x_1, \ldots, x_{n-1}, x_n\rangle \in K\}$, and similarly for A and B. Let f_{K,x_n} be the function with domain K_{x_n} such that $f_{K,x_n}(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, x_n)$, and similarly for f_A, r, and J^r. Since $A_{x_n} = r_{x_n}[B_{x_n}]$, it is readily seen to be Jordan measurable and compact. Now, $\int_A f_A = \int_K f_K = \int_{[l_B,u_B]} \int_{K_{x_n}} f_{K,x_n}$ by theorem 8.13. This equals $\int_{[l_B,r_B]} \int_{A_{x_n}} f_{A,x_n}$, which equals $\int_{[l_B,r_B]} \int_{B_{x_n}} f_{A,x_n}(r_{x_n}(x))$ $|J^{r,x_n}(x)|\,dx$, by the induction hypothesis and the fact that $r_n^{\partial i} = \delta_{in}$. By theorem 8.13 this equals $\int_B f_A(r(x))|J^r(x)|\,dx$.

5. In the notation of fact 3, $T(q, B, f_A)$. To see this, let $g = f_A \circ r$, and for $w \in \mathcal{R}^{n-1}$ let $g_w(x_n) = g(w_1, \ldots, w_{n-1}, x_n)$. Using other notation similar to that used in fact 4, and noting that $q_j^{\partial i} = \delta_{ij}$ for $j < n$, $\int_B g = \int_J \int_{[l_B,u_B]} g_w \int_J \int_{[l_C,u_C]} g_w(q_w(x))|J^{q,w}(x)|\,dx = \int_{\hat{C}} g(q(x))|J^q(x)|\,dx$.

6. In the notation of fact 3, $T(\hat{p} \restriction P^{(in)}[C], A, f_A)$. This follows by facts 4, 5, and 1.

7. In the notation of fact 3, $T(p \restriction U_C, A, f_A)$. This follows by facts 6, 2, and 1.

To complete the proof, choose a C for each $u \in p^{-1}[D]$. Take a finite subcover. Take the corresponding H-cells. Split these at their cutting planes and discard redundant H-cells. The theorem follows by fact 7 and finite additivity. QED.

The hypothesis that D be compact can be removed, provided f is required to be bounded. A proof can be found in [Spivak]; it requires the use of "partitions of unity", which have a variety of uses in extending a "local" result to a "global" one.

Additional material.

Let $\bar{B}_r^n$ denote the closed ball B_{0r}^{cl} in $\mathcal{R}^n$. That this is Jordan measurable may be seen inductively as follows. $\bar{B}_r^1$ equals $[-r, r]$. For $n > 1$ $\{x \in \bar{B}_r^n : x_n = 0\}$ is a copy of $\bar{B}_r^{n-1}$. The function $h(x_1, \ldots, x_{n-1}) = \sqrt{x_1^2 + \cdots + x_{n-1}^2}$ is continuous; by corollary 8.10 and theorem 8.7 the "upper half" of $\bar{B}_r^n$ is Jordan measurable, whence $\bar{B}_r^n$ is.

Write $b_n(r)$ for $\mu(\bar{B}_r^n)$. Using theorem 8.7, it follows that $b_1(r) = 2r$, and for $n > 1$ $b_n(r) = \int_{-r}^{r} b_{n-1}(r - x)\,dx$. From this, it follows using only linear substitution in one dimension that for $n \geq 1$, $b_n(r) = v_n r^n$ where $v_n = I_1 \cdots I_n$ and $I_n = \int_{-1}^{1}(\sqrt{1-x^2})^{n-1}\,dx$. This is clear for $n = 1$; and inductively, letting $w = x/r$, $b_n(r) = \int_{-r}^{r} c_{n-1}(\sqrt{r^2-x^2})^{n-1}\,dx = v_{n-1}r^n \int_{-1}^{1}(\sqrt{1-w^2})^{n-1}\,dw = v_{n-1}I_n r^n$.

Using integration by parts, it follows that $I_n = ((n-1)/n)I_{n-2}$ (exercise A1). Using induction, it follows that $v_n = \pi^{n/2}/(n/2)!$ if n is even, and $v_n = (2^n((n-1)/2)!\pi^{n/2})/n!$ if n is odd (exercise A2).

Exercises.

1. Show that if d is the metric function of a metric space X then $d : X \times X \mapsto \mathcal{R}$ is continuous.

2. Prove lemma 1. Hint: $\{x : |x| = 1\}$ is compact, and $|x|$ is a continuous function, so the sup is finite. Clearly $|f|_s \geq 0$. $|f|_s = 0$ iff $f(x) = 0$ for all x. $|cf|_s = |c||f|_s$ because $|(cf)(x)| = |c||f(x)|$. $|f+g|_s \leq |f|_s + |g|_s$ because $|(f+g)(x)| \leq |f|(x) + |g(x)|$. Letting f denote indifferently the matrix, $|f|_{\max} \leq |f|_s \leq \sqrt{mn}|f|_{\max}$. $|g \circ f|_s \leq |g|_s|f|_s$ because $|gf(x)| \leq |g||f||x|$.

3. Give details for the proof of lemma 6.

4. Show the following. Suppose $W \subseteq \mathcal{R}^n$ is open, $q : W \mapsto \mathcal{R}^n$ is differentiable, and $|q'(x)|_s \leq b$ for $x \in W$. Then for $x, y \in W$, $|q(y) - q(x)| \leq b|y-x|$. Hint: Let $g(\xi) = f(x + \xi(y-x))$ for $\leq \xi \leq 1$. Using the hypothesis, $|g'(\xi)| \leq b|y-x|$. Using lemma 4.10, $|g(1)-g(0)| \leq b|y-x|$.

5. Show that if $E \in \mathcal{E}$ and $\nu(E) < \delta$ then there is an $E' \in \mathcal{E}$ with $E \subseteq E'$, $\nu(E') < \delta$, and where the cells of E' are cubical of width w. Hint: If N is the number of cells C in E, expand each C so that all the l_i and u_i are in $\mathcal{Q}$, increasing the volume by less than $(\delta - \nu(E))/N$. Let w equal $1/m$ where m is a common multiple of all the denominators. Discard repeated cells.

6. Provide further details for the proof of fact 1 of theorem 13. Hint: $(f \cdot g \circ h)(x) = f(h(x)) \cdot g(h(x)) = ((f \circ h) \cdot (g \circ h))(x)$. Also, $|J^{p_1}(p_2(x))|\,|J^{p_2}(x)| = |\det(p_1'(p_2(x)))| \cdot |\det(p_2'(x))| = |\det(p_1'(p_2(x)) \cdot p_2'(x))| = |\det((p_1 \circ p_2)'(x)| = |J^p(x)|$.

A1. Show that $I_n = (n-1)(I_{n-2} - I_n)$. Hint: Write α for $\sqrt{1-w^2}$. Then $\int_{-1}^{1}\alpha^{n-1}\,dw = (n-1)\int_{-1}^{1} w^2\alpha^{n-3}\,dw = (n-1)\int_{-1}^{1}(\alpha^{n-3} - \alpha^{n-1})\,dw$.

A2. Finish the proof of the formulas for v_n. Hint: By replacing the leading two factors two at a time, show that $v_n = (1/n)(1/n-2)\cdots(1/5)3^{(n-3)/2}(I_3I_2)^{(n-1)/2}I_1$ if n is odd, and $v_n = (1/n)(1/n-2)\cdots(1/4)2^{n/2-1}(I_2I_1)^{n/2}$ if n is even.

14. k-surfaces.

Curves and surfaces in $\mathcal{R}^3$ are of great importance in vector calculus and its applications. The notion of a k-surface in $\mathcal{R}^n$ for $1 \le k \le n$ is a generalization ($k = 1$ for a curve and $k = 2$ for a surface) which, as usual, provides useful perspective in developing the theory, and has applications. Applications of k-surfaces are quite numerous, and the exact definition varies with the exact requirements. For the rest of the chapter, n, k will be positive integers with $k \le n$.

In the branch of mathematics known as algebraic geometry, a k-surface is (roughly speaking) the set of solutions to $n - k$ multivariable polynomials in n variables. A definition of a k-surface as a solution set is called an implicit definition. For example, a spherical surface in $\mathcal{R}^3$ may be defined as the set of solutions to the polynomial equation $(x - c_1)^2 + (x - c_2)^2 + (x - c_3)^2 = r^2$ for constants c_1, c_2, c_3, r (this definition is sometimes useful in calculus).

More commonly in calculus, k-surfaces are defined "parametrically", that is, as $p[D]$ where $D \subseteq \mathcal{R}^k$ and $p : D \mapsto \mathcal{R}^n$. Many theorems concerning k-surfaces are stated in terms of the "parametrization" p. In order to ensure proper behavior, restrictions may be placed on p.

An example of "wild" behavior when p is required merely to be continuous is the existence of a "space filling curve", that is, a continuous function $f : [0, 1] \mapsto \mathcal{R}^2$ such that $\mathrm{Ran}(f) = [0, 1]^2$. See [wikiSpaceFil] for further discussion.

In calculus, p is frequently required to be differentiable and D is required to be open. Facts about a compact domain K can be given by requiring $K \subseteq D$ for D open and considering $p \restriction K$. Additional restrictions of interest include that p be C_r for some r; that p be injective; that $p'(x)$ have full rank for $x \in D$; or that D be connected. Usage varies between authors. For example, when $k = 1$ the definition of a curve usually requires that D be a closed interval; if p is injective the curve is called either simple, or an arc.

For brevity, the following terminology is introduced. A C_1 k-surface in $\mathcal{R}^n$ is a function $p : D \mapsto \mathcal{R}^n$, where $D \subseteq \mathcal{R}^k$ is an open subset and p is C_1. A regular k-surface is a C_1 k-surface p, such that p is injective and $p'(x)$ has full rank for every $x \in D$. In this chapter, k-surfaces will all be regular.

Some of the proofs in the chapter are somewhat technical, and can be skipped by the reader temporarily. Some additional facts from linear algebra will be introduced, which have been left until this point for the sake of the length of chapter 4. The following is a basic and useful fact about matrix rank.

Lemma 1. If A is an $l \times m$ matrix and B is an $m \times n$ matrix then the rank of AB is less than or equal to the rank of A and the rank of B.

Proof: A linear combination of the rows of A equals xA where x is a length l row vector. If this equals 0 then so does xAB. The argument for B is similar, using rows. QED.

A matrix is said to be square if it has the same number of rows as columns. As a corollary of lemma 1, if B is square and invertible then the rank of AB equals the rank of A and if A is square and invertible then then the rank of AB equals the rank of B. (Proof: The rank of A equals the rank of ABB^{-1}, which is less than or equal to the rank of AB.)

The matrix A^TA is called the Gram matrix of A. As another corollary, if A is $l \times m$ where $l \geq m$, A^TA is invertible iff A has full rank.

Theorem 2. Suppose $p : D \mapsto \mathcal{R}_n$ is a regular k-surface, $E \subseteq \mathcal{R}^k$ is open, and $q : E \mapsto D$ is a C_1 diffeomorphism. Then $p \circ q : E \mapsto \mathcal{R}^n$ is a regular k-surface, with $p \circ q[E] = p[D]$.

Proof: By facts from earlier chapters, $p \circ q$ is C_1 and injective, and $p \circ q[E] = p[D]$. That $(p \circ q)(x)$ for $x \in E$ has full rank follows by the chain rule, lemma 13.5, and lemma 1. QED.

Theorem 3. Suppose for $l = 1, 2$, that $p_l : D_l \mapsto \mathcal{R}^n$ is a regular k-surface. Suppose $p_1[D_1] = p_2[D_2]$. Then there is a C_1 diffeomorphism q from D_1 to D_2, such that $p_1 = p_2 \circ q$.

Proof: Let $\pi : \mathcal{R}^n \mapsto \mathcal{R}^k$ be the function where $\pi_i(w) = w_i$ for $1 \leq i \leq k$. Let $\mu : \mathcal{R}^k \mapsto \mathcal{R}^n$ be the function where $\mu_i(w) = w_i$ if $1 \leq i \leq k$, and $\mu_i(w) = 0$ otherwise.

Given a regular k-surface $p : D \mapsto \mathcal{R}^n$, and a point $x \in D$, let $I = \{i_1, \ldots, i_k\}$ be a set of indices such that these rows of the $n \times k$ matrix $p'(x)$ are linearly independent. Let U be an open subset of D such that these rows of $p'(w)$ are linearly independent for $w \in U$. Let $i_{k+1}, \ldots, i_n$ be the increasing enumeration of I^c. Let $\hat{p} : U \times \mathcal{R}^{n-k} \mapsto \mathcal{R}^n$ be the function where $\hat{p}_{i_j}(w) = p_{i_j}(\pi(w))$ if $1 \leq j \leq k$, and $\hat{p}_{i_j}(w) = p_{i_j}(\pi(w)) + w_{i_j}$ otherwise. It is readily verified that $\hat{p}$ is C_1, injective, and nonsingular; by corollary 13.4, $\hat{p}$ is a C_1-diffeomorphism. By construction, $p(w) = \hat{p}(\mu(w))$ for $w \in U$.

Now, there is a unique bijection q such that $p_1 = p_2 \circ q$, namely, that where, if $p_1(x_1) = p_2(x_2)$ then $q(x_1) = x_2$. Given $x_1 \in D$ let $x_2 = q(x_1)$; for $l = 1, 2$, let U_l and $\hat{p}_l$ be constructed from p_l as above, with $x_l \in U_l$. Let $\tilde{p}_l = \hat{p}_l \restriction \hat{p}_l^{-1}[\hat{p}_1[U_1\mathcal{R}^{n-k}] \times [\hat{p}_2[U_2\mathcal{R}^{n-k}]]$. Then $\hat{q}$ is a C_1 diffeomorphism; asm μ and π are C_1, it follows that q is C_1 at any $x_1 \in D_1$; likewise q^{-1} is C_1 and any $x \in D_2$. QED.

An important use of parameterized k-surfaces is the definition of the length of a curve, the area of a surface, or in general the k-volume of a k-surface. Note that this is not the Jordan measure of $p[D]$; if $k < n$ this will usually be 0. For example it is 0 for the boundary of a Jordan measurable subset of $\mathcal{R}^n$, such as the surface of a sphere.

To determine a reasonable notion of the k-volume, first consider the case that p is a linear transformation. Say that a matrix T is upper triangular if $T_{ij} = 0$ for $i > j$. Recall the definition of an orthogonal set of vectors from chapter 7.

Lemma 4. Suppose P is an $n \times k$ matrix of rank k (in particular $k \leq n$).

a. There are an $n \times k$ matrix O_1 with orthonormal columns, and a $k \times k$ upper triangular matrix T_1, such that $P = O_1 T_1$.
b. There are an $n \times n$ orthogonal matrix O, and a $n \times k$ upper triangular matrix T, such that $P = OT$.

Proof: Suppose $v_1, \ldots v_k$ are the columns of P. Vectors $v_1', \ldots v_i'$ will be defined inductively such that $\{v_1', \ldots v_i'\}$ is an orthonormal set whose span is the same as the span of $\{v_1, \ldots v_i\}$. Let $v_1' = v_1/|v_1|$. Inductively, for $i > 1$ let $v_i'' = v_i - \sum_{j<i}(v_i * v_j')v_j'$; v_i'' cannot be 0 because v_i is not in the span of $\{v_1', \ldots v_{i-1}'\}$. Straightforward computations show that v_i'' is orthogonal to v_j' for $j < i$. Let $v_i' = v_i''/|v_i''|$. It is readily seen using the induction hypothesis that the spans of $\{v_1', \ldots v_i'\}$ and $\{v_1, \ldots v_i\}$ are equal.

Let O_1 be the matrix with columns $v_1', \ldots, v_k'$. From the construction, v_i equals $\sum_{j \leq i} t_{ij} v_j'$ for some scalars t_{ij}. Let T be the matrix where $T_{ij} = t_{ji}$ if $i \leq j$, else 0. This proves part a.

For part b, by theorem 3.3 P can be extended to a rank n $n \times n$ matrix P_1 by adding columns. Let $P_1 = O_1 T_1$ as in part a. Then $P = OT$ where $O = O_1$ and T is T_1 with columns $k+1$ through n deleted. QED.

Suppose $D \subseteq \mathcal{R}^k$ is Jordan measurable, and $p : \mathcal{R}^k \mapsto \mathcal{R}^n$ is a linear transformation with matrix P. Write P as OT as in theorem 5.b, and write T as MT_1 where $M_{ij} = 1$ if $i = j$, else 0; and T_1 is the top k rows of T. T_1 is a linear transformation of $\mathcal{R}^k$, so $\mu(T_1[D]) = |\det(T_1)|\mu(D)$ (theorem 8.6). M maps $\mathcal{R}^k$ to the subspace S of $\mathcal{R}^n$ generated by the first k standard unit vectors, in such a way that $\mathcal{R}^k$ and S are "geometrically identified". Since O is orthogonal, $MT_1[D]$ and $P[D]$ are gometrically identified also. Thus, the k-volume of $P[D]$ should equal $|\det(T_1)|\mu(D)$. It is readily verified that $|\det(T_1)| = \sqrt{\det(T_1^T T_1)}$ and $T_1^T T_1 = P^T P$, so the k-volume of $P[D]$ should equal $\sqrt{\det(P^T P)}\mu(D)$.

Suppose $p : U \mapsto \mathcal{R}_n$ is a regular k-surface, $D \subseteq U$ is Jordan measurable, and $p' \restriction D$ is bounded. In view of the preceding, the k-volume of p is defined to equal $\int_D \sqrt{\det(p'(x)^T p'(x))}\, dx$. The following theorem shows that under suitable restrictions this value depends only on $p[D]$.

Lemma 5. Suppose $p_1 : U_1 \mapsto \mathcal{R}_n$ is a regular k-surface, $D_1 \subseteq U_1$ is Jordan measurable, and $p_1' \restriction D_1$ is bounded. Suppose $U_2 \subseteq \mathcal{R}^k$ is open.

and $q : U_2 \mapsto U_1$ is a C_1 diffeomorphism. Let $p_2 = p_1 \circ q$ and let $D_2 = q^{-1}[D_1]$. Suppose $q' \restriction D_2$ is bounded. Then D_2 is Jordan measurable, and $\int_{D_1} \sqrt{\det(p_1'(x)^T p_1'(x))}\, dx$ equals $\int_{D_2} \sqrt{\det(p_2'(x)^T p_2'(x))}\, dx$.

Proof: D_2 is Jordan measurable by lemma 13.9. By theorem 13.13, $\int_{D_1} \sqrt{\det(p_1'(x)^T p_1'(x))}\, dx$ equals
$\int_{q^{-1}[D_1]} \sqrt{\det(p_1'(q(x))^T p_1'(q(x)))}\, |\det(q'(x)|\, dx$. This equals
$\int_{q^{-1}[D_1]} \sqrt{\det(q'(x)^T p_1'(q(x))^T p_1'(q(x)))(q'(x)}\, dx$, which equals
$\int_{D_2} \sqrt{\det(p_2'(x)^T p_2'(x))}\, dx$. QED.

Theorem 6. Suppose for $i = 1, 2$ that $p_i : U_i \mapsto \mathcal{R}_n$ is a regular k-surface, $D_i \subseteq U_i$ is compact and Jordan measurable, and $p_1[D_1] = p_2[D_2]$. Then $\int_{D_1} \sqrt{\det(p_1'(x)^T p_1'(x))}\, dx$ equals $\int_{D_2} \sqrt{\det(p_2'(x)^T p_2'(x))}\, dx$.

Proof: First, if $h : X \mapsto Y$ is a homeomorphism and $X' \subseteq X$ then $h \cap (X' \times h[X'])$ is a homeomorphism from X' to $h[X']$; this follows by theorem 2.1. Second, suppose $h : X \mapsto Y$ is a bijection between topological spaces, and for each $x \in X$ there is an open subset $U_x \subseteq X$ such that $x \in U_x$, $h[U_x]$ is open, and $h \cap (U_x \times h[U_x])$ is a homeomorphism. Then h is a homeomorphism (exercise 1). Third, if $p : U \mapsto \mathcal{R}_n$ is a regular k-surface, then p is a homeomorphism to $p[U]$. This follows by the existence of the functions $\hat{p}$ in the proof of theorem 3, and the first two facts.

By the foregoing, $p_i \cap (D_i \times p_i[D_i])$ is a homeomorphism, so the unique bijection $q : D_2 \mapsto D_1$ is. By theorem 3, $q \cap (D_2^{\text{int}} \times q[D_2^{\text{int}}])$ is a C_1-diffeomorphism. So far compactness of D_i has not been used; if D_1 and D_2 are compact (for which it suffices that either is) then q' is bounded. Then, using lemma 5 and obvious notation, $\int_{D_1} = \int_{D_1^{\text{int}}} = \int_{D_2^{\text{int}}} = \int_{D_2}$. QED.

In fact, the integral depends only on $p[D]$ under more general restrictions. For example, the "area theorem" of higher analysis implies that $\mu_k(D)$ equals the "k-dimensional Hausdorff measure" of $p[D]$ under more general restrictions. See theorem 5.1.1 of [KrantzParks]; and also exercise 12 of [TaoHausDim].

Theorem 6 illustrates that for some purposes the parameterization p is only a convenience for specifying $p[D]$, the set of points of the k-surface. There are some contexts in which p is an object of interest, though, for example if $k = 1$, $n = 3$, and $p(t)$ is the position of a particle in space at the time t. $p[D]$ is the curve traced by the point, and of course there are contexts where this is the object of interest.

Let M_k denote the family of subsets $S \subseteq \mathcal{R}^n$ such that there is a regular k-surface $p : U \mapsto \mathcal{R}_n$, and a Jordan measurable compact subset $D \subseteq U$, such that $S = p[D]$. For $S \in M_k$ let $\mu_k(S)$ equal $\int_D \sqrt{\det(p'(x)^T p'(x))}\,dx$.

Theorem 7.

a. M_k is closed under translation and linear transformation.
b. If $S \in M_k$ and t is a translation then $\mu_k(t[S]) = \mu_k(S)$.
c. If $S \in M_k$ and l is a linear transformation then $\mu_k(l[S]) = \det(l)\mu_k(S)$.

Proof: Let t be a translation and l a linear transformation. If p is a regular k-surface then so are $t \circ p$ and $l \circ p$. If $p : U \mapsto \mathcal{R}^n$ and $D \subseteq U$ is compact and Jordan measurable then so are $t[D]$ and $l[D]$. Part a follows. Part b follows by theorem 8.5. Part c follows by theorem 8.6. QED.

The function μ_k is a correct notion, but its technical restrictions can make its use unwieldy. For example a notion of finite additivity is unwieldy. Say that S is the almost disjoint union of $S_1, S_2 \subseteq S$ iff $S = S_1 \cup S_2$ and $S_1 \cap S_2 \subseteq S_1^{\rm bd}, S_2^{\rm bd}$.

Theorem 8. If $S, S_1, S_2 \in M_k$ and S is the almost disjoint union of S_1, S_2 then $\mu_k(S) = \mu_k(S_1) + \mu_k(S_2)$.

Proof: Let $D_i = p^{-1}[S_i]$. S_i is a closed subspace of S, so D_i is closed, and it is clearly bounded, so D_i is compact. By observations in the proof of theorem 6, and lemma 13.6, $p[D_i^{\rm bd}] = S_i^{\rm bd}$. It follows that $\int_{D_2 - D_1} = \int_{D_2}$, and the theorem follows. QED.

The surface $|x - c| = r$ in $\mathcal{R}^{n+1}$ is called an "n-sphere" or "n-hypersphere". For any n-sphere S, $\mu_n(S)$ may be defined as the sum of $\mu_n(S_i)$ where $\{S_i\}$ is a finite family of elements of M_n and S is the almost disjoint union of the S_i. Let S_n denote the n-sphere where $c = 0$ and $r = 1$. Let s_n denote $\mu_n(S_n)$. Then $\mu_n(S)$ equals $s_n r^n$. See the additional material for the value of s_n.

While regular k-surfaces are clearly useful, some subspaces which are clearly k-surfaces cannot be desribed as such. The n-sphere $S_n = \{x \in \mathcal{R}^{n+1} : |x| = 1\}$ is an example of such. It is compact, so cannot be homeomorphic to an open subset of $\mathcal{R}^n$ (although $S_n - p$ is for a point $p \in S_n$). As just seen, even defining the n-volume of S_n requires ad-hoc methods. To circumvent this problem, k-surfaces may be defined which are subspaces covered by "patches" which are parameterized k-surfaces.

k-surfaces of this type are called k-manifolds. These are of interest in topology. A topological k-manifold is defined to be a suitable topological space X, such that for each point $x \in X$, there is an open subset $U \subseteq X$ with $x \in U$, and a homeomorphism $p : V \mapsto U$ where V is an open subset of $\mathcal{R}^k$, What consitutes a suitable space X varies from author to author. Some authors require that it be "metrizable", others that it satisfy the slightly weaker requirement "paracompact Hausdorff", etc.

Restrictions may be placed on topological k-manifolds. In one commonly encountered set of restrictions, X is a subspace of some $\mathcal{R}^n$ where $n \geq k$, and for each $p : V \mapsto U$, p is (at least) C_1 and $p'(x)$ has full rank for each $x \in V$. (The last restriction is not redundant; see exercise 2). See theorem 5.2 of [Spivak] for an equivalent formulation of these restrictions. $S_n \subseteq \mathcal{R}^{n+1}$ may be seen to be an n-manifold satisfying them (exercise 3).

Additional material.

S_n has a useful parametrization, known as "hyperspherical coordinates". Namely, given angles $\theta_1, \ldots, \theta_n$, let $x_1 = \prod_{j=1}^n \cos(\theta_j)$ and for $2 \leq i \leq n+1$ let $x_i = \sin(\theta_{i-1}) \prod_{j=i}^n \cos(\theta_j)$. For $x \in S_n$, i.e., $x \in \mathcal{R}^{n+1}$ with $|x| = 1$, these angles can be determined by successive projection. Let θ_n be such that $x_{n+1} = \sin(\theta_n)$. If $n > 1$ let y be the projection of x on the hyperplane normal to the e_{n+1}. The length of y is $\cos(\theta_n)$; the remaining angles may be obtained inductively.

This parameterization p is clearly C_1. To make it injective, consider p restricted to the domain $D_n = (-\pi/2, \pi/2)^n$. It is readily verified inductively that $p[D_n] = \{x \in S_n : x_1 > 0\}$, and p is injective.

Direct computation shows that the Gram matrix G_n of $p'(x)$ is diagonal, with $G_{n,ii}$ equalling $\prod_{j=i+1}^n \cos(\theta_j)^2$ (exercise A1). On the domain D_n, G_n is invertible, and p' has full rank.

Let J_n denote $\prod_{i=2}^n \cos(\theta_i)^{i-1}$, which is $\sqrt{\det(G_n)}$. Let $C_n = \int_{-\pi/2}^{\pi/2} \cos(\theta_i)^{i-1}\, d\theta_i$. J_n is bounded on D_n^{cl}, so $\int_{D_n} J_n = \int_{D_n^{\mathrm{cl}}} J_n = \prod_{i=1}^n C_i$; let H_n denote this quantity. It may be verified that $\mu_n(S_n) = 2H_n$ (exercise A2). It follows using the foregoing and observations from the additional material of chapter 13 that $s_{n-1} = nv_n$ (exercise A3).

Exercises.

1. Suppose $h : X \mapsto Y$ is a bijection between topological spaces, and for each $x \in X$ there is an open subset $U_x \subseteq X$ such that $x \in U_x$, $h[U_x]$ is open, and $h \cap (U_x \times h[U_x])$ is an open map. Show that h is an open map. Hint: Suppose V is open. Using lemma 5.1, $f[V] \cap f[U_x]$ is an open subset of Y.

2. Construct a C_1 map $p : (-1, 1) \mapsto \mathcal{R}^2$ which is a homeomorphism to its range considered as a subspace, but where $p'(0)$ does not have full rank. Hint: let $p(x) = (x^3, 0)$.

3. Construct a C_1 injective map $p : \mathcal{R}^n \mapsto S_n - \{1, 0, \ldots, 0\}$ such that $p'(x)$ has full rank for every x. Hint: consider the map $\langle x_1, \ldots, x_n, x_{n+1}\rangle \mapsto (1/x_{n+1})\langle x_1, \ldots, x_n\rangle$.

A1. Prove that the Gram matrix of the derivative matrix of the hyperspherical coordinate parametrization is as stated. Hint: Write an explicit expression for the derivative matrix. G_{ij} equals the inner product of columns i and j.

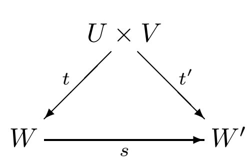
A2. Show that $\mu_n(S_n) = 2H_n$. Hint: Let $S_n^{\geq\xi} = \{x \in S_n : x_1 \geq \xi\}$. It suffices to show that $\mu_n(S_n^{\geq 0}) = H_n$. Let $S_n^{(\eta)} = p[(-\pi/2+\eta, \pi/2-\eta)^n]$. Since J_n is bounded, and for all ξ there is an η such that $S_n^{\geq\xi} \subseteq S_n^{(\eta)}$, $H_n = \lim_{\xi\to 0}\mu(S_n^{\geq\xi})$. $\mu_n(S_n^{\geq 0})$ also equals this limit.

A3. Show that $s_{n-1} = nv_n$. Hint: First show by induction that $C_n = I_{n-1}$ for $n > 1$, where I_n is as in chapter 13. Then inductively $s_{n-1} = C_{n-1}s_{n-2} = C_{n-1}(n-1)v_{n-1} = I_{n-2}(n-1)v_{n-1} = nI_{n-2}((n-1)/n)v_{n-1} = nI_nv_{n-1} = nv_n$.

15. Tensors.

In abstract algebra, the "tensor product" in the "category of R-modules" for a commutative ring R is defined. This is a fundamental construction of modern abstract algebra. Specializing to vector spaces over a field F results in simplifications. In vector calculus, of course, the case of interest is finite dimensional vector spaces over $\mathcal{R}$. The rank k tensors of applied mathematics are elements of the tensor product of k copies of the 3-dimensional vector space $\mathcal{R}^3$, which as will be seen is a copy of $\mathcal{R}^{3^k}$ (and similary for tensors over $\mathcal{R}^4$).

Suppose U and V are vector spaces over a field F. Recall that their Cartesian product $U \times V$ has as elements the ordered pairs $\langle u, v\rangle$ with $u \in U$, $v \in V$. A vector space W over F, together with a bilinear map $t : U \times V \mapsto W$, is said to be a tensor product for U and V if the following holds. Given a vector space W' over F, together with a bilinear map $t' : U \times V \mapsto W'$, there is a unique linear map $s : W \mapsto W'$ such that $t' = t \circ s$. The requirement on t is summarized in the following "commutative diagram".



The preceding definition is an example of a "category-theoretic" definition; however any foray into category theory is unnecessary and all the desired properties of the tensor product can be proved immediately. Of course, these are examples of category-theoretic properties which follow by general facts.

Theorem 1. The tensor product is essentially unique. Indeed, given two such W_1, t_1 and W_2, t_2, the unique linear maps $s_{12} : W_1 \mapsto W_2$ and $s_{21} : W_2 \mapsto W_1$ are functions inverse to each other.

Proof: The identity function $\iota : W_1 \mapsto W_1$ satisfies the requirements of theorem 1, and so does $s_{21} \circ s_{12}$, so they are equal. Similarly $s_{12} \circ s_{21} = \iota$. QED.

Theorem 2. U and V have a tensor product. Indeed, let B be a basis for U and let C be a basis for V. Let W be the set of "formal" linear combinations of "basis elements" $b \otimes c$ where $b \in B$ and $c \in C$. Let t be the bilinear map such that $t(b, c) = b \otimes c$.

Proof: Given W' and t', let $s(b \otimes c) = t'(b, c)$. By definition this must be the case; and by linearity, s extends uniquely to a linear map on W. QED.

The notation $U \otimes V$ is used to denote the tensor product of two vector spaces. For $u \in U$ and $v \in V$, $u \otimes v$ may be used to denote $t(u, v)$; if $u = \sum_i u_i b_i$ and $v = \sum_j v_i c_j$ then by bilinearity, $u \otimes v = \sum_{ij} u_i v_j \, b_1 \otimes c_j$.

If U is n-dimensional and V is m-dimensional then $U \otimes V$ is nm-dimensional. From hereon only this case will be considered. An element of the form $u \otimes v$ is called a dyad; clearly not all elements of $U \otimes V$ are of this form.

Tensors have several alternative characterizations. For one example, consider the case where U is F^n and b_i is the standard unit vector e_i, and V is F^m and c_j is the standard unit vector e_j; $b_i \otimes c_j$ may be taken to be the matrix which is 0, except in the ij position, where it is 1. $U \otimes V$ may be considered to be the $n \times m$ matrices. The matrix of a dyad $u \otimes v$ is obtained by taking the "outer product" of the vectors u and v.

The tensor product $V_1 \otimes \cdots \otimes V_k$ of several factors may be "officially" defined in various ways, for examples as the right associated iteration of the binary tensor product, $V_1 \otimes (\cdots \otimes (V_{k-1} \otimes V_k) \cdots)$. There is a standard isomorphism between $V_1 \otimes (V_2 \otimes V_3)$ and $(V_1 \otimes V_2) \otimes V_3$, mapping $v_1 \otimes (v_2 \otimes v_3)$ to $(v_1 \otimes v_2) \otimes v_3$. This generalizes to any two methods of associating $V_1 \otimes \cdots \otimes V_k$. Basis elements of $V_1 \otimes \cdots \otimes V_k$ may be written as $b_{1i_1} \otimes \cdots \otimes b_{ki_k}$ where b_{ji_j} is an element of a chosen basis for V_j. It is easily seen that the map determined by $\langle b_{1i_1}, \ldots, b_{ki_k} \rangle \mapsto b_{1i_1} \otimes \cdots \otimes b_{ki_k}$ is multilinear (exercise 2), in fact satisfies the defining property (suitably generalized) of the map t in the definition of the binary tensor product.

If V is an n-dimensional vector space over the field F, let $V^{\otimes k}$ denote the tensor product of V with itself k times. The elements of $V^{\otimes k}$ are the tensors of rank k. Letting $\{b_i : 1 \leq i \leq n\}$ be a basis for V, such a tensor u may be written as $\sum_{i_1 \ldots i_k} u_{i_1 \ldots i_k} b_{i_1} \otimes \cdots \otimes b_{i_k}$ where

$1 \le i_l \le n$ for $1 \le l \le k$. In particular, $V^{\otimes k}$ has dimension n^k. The scalars $u_{i_1 \ldots i_k}$ are known as the components of the tensor.

Various facts concerning tensors may be considered in category-theoretic terms, which has the advantage of generality. However, for routine applications it may be easier to consider such facts in terms of the components of the tensors. This might seem to be an inferior approach, in that the components depend on the choice of bases. However, these are related for different choices of bases, via the "tensor transformation law".

Let δ_{ij} equal 1 if $j = i$, else 0 (this very useful function goes by the name of the Kronecker delta function). Suppose $\{b_i\}, \{b'_i\}$ are two bases for the n-dimensional vector space U over the field F. Then $b'_i = \sum_j d_{ij} b_j$ for some elements d_{ij}, and $b_i = \sum_j \bar{d}_{ij} b'_j$ for some elements $\bar{d}_{ij}$. Substituting the second expression in the first and simplifying yields $\sum_k d_{ik} \bar{d}_{kj} = \delta_{ij}$; similarly $\sum_k \bar{d}_{ik} d_{kj} = \delta_{ij}$. In particular, by linear algebra the matrix $\bar{d}_{ij}$ is uniquely determined from the matrix d_{ij}.

If $u = \sum_i u_i b_i$ then by substituting the expression for b_i, $u = \sum_j (\sum_i \bar{d}_{ij} u_i) b'_j$. Writing u'_j for the components of u in the "primed" basis, $u'_j = \sum_i \bar{d}_{ij} u_i$. The components of u transform "contravariantly" to the expression giving the elements of the new basis in terms of the old (in matrix terms, via the inverse transpose). Thus, a vector (rank 1 tensor) is an n-tuple of elements of F which transform contravariantly under a change of basis.

By multilinearity, $b'_{i_1} \otimes \cdots \otimes b'_{i_k} = \sum_{j_1 \ldots j_k} d_{i_1 j_1} \cdots d_{i_k j_k} b_{j_1} \otimes \cdots \otimes b_{j_k}$. This may be used to derive the transformation law for the components. namely, $u'_{j_1 \ldots j_k} = \sum_{i_1 \ldots i_k} \bar{d}_{i_1 j_1} \cdots \bar{d}_{i_k j_k} u_{i_1 \ldots i_k}$ (exercise 1). The transformation law for more general tensor products may similarly be determined, requiring additional notational complexity.

Since $V^{\otimes k}$ is a vector space, the operations of addition and scalar multiplication are defined on it. To add two rank k tensors, obtain their components with respect to some basis, and add correponding components. The operation "commutes" with component transformation, so the result is independent of the choice of basis. Of course, in the case of tensors, the components are indexed by a vector of integers rather than a single integer; but this does not change anything in the case of addition or scalar multiplication.

The operation $\otimes$ mapping $V^{\otimes k} \times V^{\otimes l}$ to $V^{\otimes k+l}$ may be defined by its action on the elements of a basis, namely, $(b_{i_1} \otimes \cdots \otimes b_{i_k}) \otimes (b_{j_1} \otimes \cdots \otimes b_{j_l}) = b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes b_{j_1} \otimes \cdots \otimes b_{j_l}$. This is just an application of the standard isomorphism from $V^{\otimes k} \times V^{\otimes l}$ to $V^{\otimes k+l}$.

It is convenient to allow $k = 0$; $V^{\otimes 0}$ is defined to be F, and the basis may be taken as 1. Addition is just addition in F, and $1 \otimes (b_{j_1} \otimes$

$\cdots \otimes b_{j_l}) = (b_{j_1} \otimes \cdots \otimes b_{j_l}) \otimes 1 = b_{j_1} \otimes \cdots \otimes b_{j_l}$.

The maps $\otimes : V^{\otimes k} \times V^{\otimes l} \mapsto V^{\otimes k+l}$ satisfy the associative law (exercise 3). (The "direct sum" operation may be used to combine the vector spaces $V^{\otimes k}$ for various k into a single structure, called the "tensor algebra" over V; in this setting $\otimes$ is a single associative operation.) They clearly do not satisfy the commutative law; indeed, for distinct i_1, i_2, $b_{i_1} \otimes b_{i_2}$ and $b_{i_2} \otimes b_{i_1}$ are distinct basis elements of $V^{\otimes 2}$.

For some tensors (for example $b_{i_1} \otimes b_{i_1}$) the indices may be permuted without changing the value. These turn out to be of interest. Given a tensor $u \in V^{\otimes k}$ with components $u_{i_1 \ldots i_k}$, and a permutation $\pi \in \mathrm{Sym}_k$, let u^π denote the tensor with components $u_{\pi(i_1)\ldots\pi(i_k)}$; u is said to be symmetric if $u^\pi = u$ for all $\pi \in \mathrm{Sym}_k$. It is easy to see that if the components are symmetric in one basis then they are symmetric in any basis (exercise 4).

The family of symmetric tensors is closed under the operations of addition and scalar multiplication, so forms a vector subspace of $V^{\otimes k}$, which will be denoted $V^{\otimes_S k}$. $V^{\otimes_S 0}$ is defined, and equals F. Let $b^S_{i_1,\ldots,i_k}$ denote $\sum_{\pi \in \mathrm{Sym}_k} b_{\pi(i_1)} \otimes \cdots \otimes b_\pi(i_1)$. These elements are symmetric, that is, $b^S_{\pi(i_1),\ldots,\pi(i_k)} = b^S_{i_1,\ldots,i_k}$ for $\pi \in \mathrm{Sym}_k$. As is easily seen they generate $V^{\otimes_S k}$. Further, they are linearly independent, since any nontrival linear dependence would yield one involving basis elements of $V^{\otimes k}$. Thus they in fact form a basis.

For $k = 2$ a symmetric tensor, considered as a matrix, is just a symmetric matrix. The basis elements are matrices with a single 2 on the diagonal, or a symmetric pair of 1's. Many authors multiply the symmetric sum by $1/k!$; but this will not be done here.

Each element $b^S_{i_1,\ldots,i_k}$ can be written in exactly ony way so that $i_1 \leq \ldots \leq i_k$. In a "multiset" of elements, an element occurs with some nonzero integer multiplicity (a more tedious formal definition is left to the reader). Clearly, a sequence $i_1 \leq \ldots \leq i_k$ where $i_j \in \{1, \ldots, n\}$ is essentially the same thing as a k element mutiset of elements from$\{1, \ldots, n\}$. This number is well-known to be

$$\binom{n+k-1}{k}$$

(exercise 5), and this is the dimension of $V^{\otimes_S k}$.

A tensor $u \in V^{\otimes k}$ is said to be antisymmetric if $u^\pi = (-1)^{\mathrm{sg}(\pi)} u$ for all $\pi \in \mathrm{Sym}_k$. As for symmetric tensors, if the components are antisymmetric in one basis then they are antisymmetric in any basis. Also, the antisymmetric tensors comprise a subspace of $V^{\otimes k}$, which will be denoted $V^{\wedge^k}$. $V^{\wedge 0}$ is defined, and equals F.

Let $b^{\wedge}_{i_1,\ldots,i_k}$ denote $\sum_{\pi\in\text{Sym}_k}(-1)^{\text{sg}(\pi)}b_{\pi(i_1)}\otimes\cdots\otimes b_\pi(i_k)$. These elements are antisymmetric, that is, $b^{\wedge}_{\pi(i_1),\ldots,\pi(i_k)} = (-1)^{\text{sg}(\pi)}b^S_{i_1,\ldots,i_k}$. They generate $V^{\wedge^k}$. They are 0 unless the i_j are distinct; in particular if $k > n$ then $V^{\wedge^k} = \{0\}$. If $k \le n$ then $\{b^{\wedge}_{i_1,\ldots,i_k} : i_1 < \ldots < i_k\}$ comprises a basis for $V^{\wedge^k}$, which has dimension $\binom{n}{k}$.

A useful and common notational device is to let $b_{i_1}\wedge\cdots\wedge b_{i_k}$ denote $b^{\wedge}_{i_1,\ldots,i_k}$. These generate $V^{\wedge^k}$. For $k = 1$ they form a basis. For $k > 1$ elements with repeated indices are 0, and the nonzero elements satisfy the linear dependencies $b_{\pi(i_1)}\wedge\cdots\wedge b_{\pi(i_k)} = (-1)^{\text{sg}(\pi)}b_{i_1}\wedge\cdots\wedge b_{i_k}$.

The vector space $V^{\wedge k}$ may alternatively be defined abstractly. The discussion benefits from some basic concepts from the algebra of vector spaces, namely, quotients by subspaces, and homomorphisms. These are commonly covered in introductory texts, and the reader is assumed to be familiar with them.

There is a unique homomorphism $\phi : V^{\otimes k} \mapsto V^{\wedge k}$, where $\phi(b_{i_1}\otimes\cdots\otimes b_{i_k})$ equals $b_{i_1}\wedge\cdots\wedge b_{i_k}$ (namely, "extend by linearity"). Its kernel K^k is by definition the set of $u \in V^{\otimes k}$ such that $\phi(u) = 0$. By standard algebra, $V^{\wedge k}$ is the quotient of $V^{\otimes k}$ by the subspace K^k. K^k may be seen to be the subspace generated by the elements $b_{i_1}\wedge\cdots\wedge b_{i_k}$ with a repeated index, and $b_{\pi(i_1)}\wedge\cdots\wedge b_{\pi(i_k)} - (-1)^{\text{sg}(\pi)}b_{i_1}\wedge\cdots\wedge b_{i_k}$ with no repeated index (exercise 6).

Analogous to the bilinear map $\otimes : V^{\otimes k}\times V^{\otimes l}\mapsto V^{\otimes k+l}$, where There is an bilinear map $\wedge : V^{\wedge k}\times V^{\wedge l}\mapsto V^{\wedge k+l}$, where $(b_{i_1}\wedge\cdots\wedge b_{i_k})\wedge(b_{j_1}\wedge\cdots\wedge b_{j_l}) = b_{i_1}\wedge\cdots\wedge b_{i_k}\wedge b_{j_1}\wedge\cdots\wedge b_{j_l}$. It suffices to show that if $v_1 - v_2 \in K^k$ and $w_1 - w_2 \in K^l$ then $v_1\wedge w_1 - v_2\wedge w_2 \in K^{k+l}$. This is left to exercise 7. The operation is known as the exterior or wedge product. It satisfies the following algebraic identities:

- bilinearity
- associativity
- $u_2\wedge u_1 = (-1)^{kl}u_1\wedge u_2$, anticommutativity

The proof is left to exercise 8.

Additional material.

Readers with some familiarity with category theory should not be surprised that the tensor product of linear transformations can be defined. Suppose $f_1 : U_1 \mapsto V_1$ and $f_2 : U_2 \mapsto V_2$ are linear transformations. Let $f_1\times f_2 : U_1\times U_2 \mapsto V_1\times V_2$ denote the map $\langle u_1, u_2\rangle \mapsto \langle f_1(u_1), f_2(u_2)\rangle$. This is readily verified to be a linear transformation between vector spaces.

Theorem A1. Suppose $f_1 : U_1 \mapsto V_1$ and $f_2 : U_2 \mapsto V_2$ are linear transformations. There is a unique linear transformation $f_1\otimes f_2 : U_1\otimes$

$U_2 \mapsto V_1 \otimes V_2$, such that $f_1 \otimes f_2 \circ t = t \circ f \times g$, that is, $f_1 \otimes f_2(u_1 \otimes u_2) = f_1(u_1) \otimes f_2(u_2)$.

Proof: It is readily seen that $t \circ f \times g$ is bilinear. The existence and uniqueness of $f_1 \otimes f_2$ follows by theorems 1 and 2. QED.

Suppose V is a vector space of dimension n over the field F. Recalling that F is a 1-dimensional space over itself, the vector space $L(V; F)$ as defined in chapter 5 is called the dual space, and its elements linear functionals. It is often denoted V^*. Given a basis $\{b_i : i \in I\}$ for V let β_i be the linear functional such that $\beta_i(b_j) = \delta_{ij}$. These are linearly independent: if $\sum_i a_i\beta_i = 0$, applying the left side to b_i yields $a_i = 0$. They are in fact a basis: given $f \in V^*$, $f = f(b_1)\beta_1 + \cdots + f(b_n)\beta_n$, since both sides have the same values on the b_i. In particular, V^* has the same dimension n as V. The basis $\{\beta_j\}$ is said to be the dual basis to $\{b_j\}$.

Suppose b'_i is another basis for V, where $b'_i = \sum_j d_{ij}b_j$. Let β'_i be the dual basis to b'_i. Then $\beta'_i(b_j) = \beta'_i(\sum_k \bar{d}_{jk}b'_k) = \sum_k \bar{d}_{jk}\delta_{ik} = \bar{d}_{ji}$; and $(\sum_k \bar{d}_{ki}\beta_k)(b_j)$ also equals $\bar{d}_{ji}$.

It follows that $\beta'_i = \sum_k \bar{d}_{ki}\beta_k$; and that the components of a linear functional, expressed as a linear combination of the dual basis elements, transform by the same expression as the basis change expression. Such a transformation of the components of a vector is said to be covariant.

In applications, rank $k + l$ tensors with components which transform contravariantly in the first k factors, and covariantly in last l, are encountered. To provide a mathematical setting for this phenomenon, the "mixed" tensor product $V^{\otimes k} \otimes V^{*\otimes l}$ is introduced. The tensors $b_{i_1} \otimes \ldots \otimes b_{i_k} \otimes \beta_{j_1} \otimes \ldots \otimes \beta_{j_l}$ comprise a basis. For elements in $V^{\otimes k} \otimes V^{*\otimes l}$, the components transform contravariantly in the first k indices, and covariantly in the last l.

The product of $b_{i_1} \otimes \ldots \otimes b_{i_k} \otimes \beta_{j_1} \otimes \ldots \otimes \beta_{j_l}$ and $b_{p_1} \otimes \ldots \otimes b_{p_r} \otimes \beta_{q_1} \otimes \ldots \otimes \beta_{q_s}$ may be defined to be $b_{i_1} \otimes \ldots \otimes b_{i_k} \otimes b_{p_1} \otimes \ldots \otimes b_{p_r} \otimes \beta_{j_1} \otimes \ldots \otimes \beta_{j_l} \otimes \beta_{q_1} \otimes \ldots \otimes \beta_{q_s}$.

Exercises.

1. Show that the map determined by $\langle b_{1i_1}, \ldots, b_{ki_k} \mapsto b_{1i_1} \otimes \cdots \otimes b_{ki_k}$ of the $k$ary tensor product is multilinear.

2. Derive the tensor transformation law from the expression for $b'_{i_1} \otimes \cdots \otimes b'_{i_k}$ given in the chapter. Hint: Replace i, j, k in the argument for vectors by $\vec{\imath}$, $\vec{\jmath}$, $\vec{l}$. Note that $\bar{d}_{\vec{\imath}\vec{\jmath}} = \bar{d}_{i_1j_1} \cdots \bar{d}_{i_kj_k}$.

3. Show that the maps $\otimes : V^{\otimes k} \times V^{\otimes l} \mapsto V^{\otimes k+l}$ satisfy the associative law. Hint: This is immediate for basis elements, and it follows for arbitrary elements by multilinearity.

4. Show that if a tensor is symmetric in one basis then it is symmetric in any basis; and similarly for antisymmetric components. Hint: In the symmetric case, using notation as in exercise 2, let $u'_{\vec{i}}$ denote the coefficient of $b'_{\vec{i}}$ in the expansion of the tensor u in the primed basis. Then $u'_{\pi(\vec{\imath})} = \sum_{\vec{\jmath}} \bar{d}_{\pi(\vec{\imath}),\vec{\jmath}} u_{\vec{\jmath}} = \sum_{\vec{\jmath}} \bar{d}_{\pi(\vec{\imath}),\pi(\vec{\jmath})} u_{\pi(\vec{\jmath})} = \sum_{\vec{\jmath}} \bar{d}_{\vec{\imath}\vec{\jmath}} u_{\pi(\vec{\jmath})} = \sum_{\vec{\jmath}} \bar{d}_{\vec{\imath}\vec{\jmath}} u_{\vec{\jmath}} = u'_{\vec{\imath}}$.

5. Show that the number of multisets of size k of elements from $\{1,\ldots,n\}$ equals $\binom{n+k-1}{k}$. Hint: It equals the number of ways of choosing $n-1$ separator positions from $n-1+k$ total positions.

6. Using notation as in exercise 2, show that K^k is the subspace generated by the elements $b_{\vec{i}}$ where $\vec{i}$ has a repeated index, and $b_{\pi(\vec{i})} - (-1)^{\text{sg}(\pi)} b_{\vec{i}}$. Hint: Given a linear combination $\sum_{\vec{i}} c_{\vec{i}} b_{\vec{i}}$ of distinct basis elements in K^k, it may be assumed that no $\vec{i}$ has a repeated index. Say that two index vectors $\vec{i}$ and $\vec{j}$ are in the same class if one is a permutation of the other; it may be assumed that the $\vec{i}$ in the linear combination are in the same class. Let s be the number of summands; then $s \geq 2$. If $s > 2$ then s can be reduced. If $s = 2$ then the linear combination is a multiple of a generating one.

7. Show that if $v_1 - v_2 \in K^k$ and $w_1 - w_2 \in K^l$ then $v_1 \wedge w_1 - v_2 \wedge w_2 \in K^{k+l}$. Hint: If $b_{\pi(i_1)} \wedge \cdots \wedge b_{\pi(i_k)} = (-1)^{\text{sg}(\pi)} b_{i_1} \wedge \cdots \wedge b_{i_k}$ then $b_{\pi(i_1)} \wedge \cdots \wedge b_{\pi(i_k)} \wedge b_{i_{k+1}} \wedge \cdots \wedge b_{i_{k+l}} = (-1)^{\text{sg}(\pi)} b_{i_1} \wedge \cdots \wedge b_{i_k} \wedge b_{i_{k+1}} \wedge \cdots \wedge b_{i_{k+l}}$. Using linearity and exercise 8, $v_1 \wedge w_1 - v_2 \wedge w_1 \in K^{k+l}$. Similarly $v_2 \wedge w_1 - v_2 \wedge w_2 \in K^{k+l}$.

8. Show that the exterior product $\wedge$ has the algebraic properties stated in the chapter. Hint: It is bilinear since it is extended by linearity from its definition on the basis elements. Associativity on basis elements follows readily, and associativity in general follows by further computation. Similarly, anticommuativity follows on basis elements (transpose each b_{j_t} successively left k times) and hence in general.

16. Differential forms.

A tensor field, more specifically a rank k tensor field (again note the name collision) on an open subset $W \subseteq \mathcal{R}^n$, is a function $f : W \mapsto (\mathcal{R}^n)^{\otimes k}$. The map is usually required to be continuous, where $(\mathcal{R}^n)^{\otimes k}$ is equipped with the metric topology of the Eucliden norm. In calculus f is typically required to be (at least) C_1. If $k = 0$ f is called a scalar field; and if $k = 1$ f is called a vector field.

A differential form, more specifically a k-form on an open subset $W \subseteq \mathcal{R}^n$, is a function $\omega : W \mapsto (\mathcal{R}^n)^{\wedge k}$ (where $0 \leq k \leq n$). Differential forms have many uses in applied mathematics. Although the definition may seem mysterious at first, increasing familiarity with their properties will provide insight into their nature. In this chapter, their algebraic properties, integration of a k-form over a k-surface, and exterior differentation, will be considerd in turn.

It is standard practice to write $dx_1, \ldots, dx_n$ for the basis elements of $(\mathcal{R}^n)^{\wedge 1}$, whence basis elements of $(\mathcal{R}^n)^{\wedge k}$ are of the form $dx_{t_1} \wedge \cdots \wedge dx_{t_k}$ where $1 \leq t_1 < \cdots < t_k \leq n$. To simplify the notation, given k with $0 \leq k \leq n$ and a set $T = \{t_1, \ldots, t_k\}$ where $1 \leq t_1 < \cdots < t_k \leq n$, let dx_T denote $dx_{t_1} \wedge \cdots \wedge dx_{t_k}$ (if $k = 1$ $T = \emptyset$ and dx_T is 1). Then for a differential form ω, $\omega = \sum_T \omega_T dx_T$ for some scalar fields ω_T. The ω_T are all C_r iff ω is. In what follows, $\omega = \sum_T \omega_T dx_T$ will often be assumed, when n and k are given, with T ranging over the $\binom{n}{k}$ k-element subsets of $\{1, \ldots, n\}$.

Fixing n and W, let Forms_k denote the space of k-forms on W. As noted in chapter 1, addition and scalar multiplication may defined pointwise on Forms_k, and equipped with these operations Forms_k becomes a real vector space. The operation $\wedge : \text{Forms}_k \times \text{Forms}_l \mapsto \text{Forms}_{k+l}$ may be defined pointwise also, and inherits properties of the operation $\otimes : V^{\otimes k} \times V^{\otimes l} \mapsto V^{\otimes (k+l)}$ given in exercise 15.8, namely, it is bilinear, associative, and anticommutative.

One further piece of notation which will be required is that for the Jacobian of the $k \times k$ matrix whose ith row is the t_ith row of p', where $p : U \mapsto \mathcal{R}^n$ for an open subset $U \subseteq \mathcal{R}^n$. This will be denoted J^p_T; it's value at a point $v \in U$ is $\det([p^{\partial_j}_{t_i}(v)])$.

Suppose ω is a k-form with domain W, p is a C_1 k-surface with domain U, $D \subseteq U$ is Jordan measurable, $S = p[D]$, and $S \subseteq W$. Then the integral $\int_S \omega$ of the form over the surface is defined by the equation

$$\int_S \omega = \sum_T \int_D (\omega_T \circ p) \cdot J^p_T.$$

Note that p is not required to be regular, although in this chapter it invariably will be.

For a helpful example, given a vector field $v : \mathcal{R}^3 \mapsto \mathcal{R}^3$, the integral over a curve of the tangential component of v, and the integral over a surface of the normal component of v, can be expressed as the integral of a differential form. Since the discussion is secondary, it is given later in the chapter Further "geometric" considerations regarding integrating differential forms may be found in [Bachmann].

Many authors (including [Bachman]) use an alternative definition for tensors, which results in different notation for various operations with differential forms. Namely, for a vector space V over a field F, let $L(V^k; F)$ denote the vector space of k-ary multilinear forms. Then $V^{\otimes k}$ may be taken to be $L(V^k; F)$ (exercise 1).

Recalling the basis $\{\beta_i\}$ for $L(V; F)$ defined in chapter 15, dx_i may be taken as the 1-form on $\mathcal{R}^n$ where $dx_i(w) = \beta_i$ for $w \in \mathcal{R}^n$, so that for a vector y (in the copy of $\mathcal{R}^n$ constituting the "tangent space", or space of possible direction vectors, at w), $dx_i(w)(y) = y_i$. Letting

$J_T(v_1,\ldots,v_n)$ denote the "Jacobian" as above for the matrix whose columns are $v_1,\ldots,v_k$, and using the definition of the antisymmetrization operation,

$\omega(x)(v_1,\ldots,v_k)=\sum_T \omega_T(x)J_T(v_1,\ldots,v_n)$.

Thus, $\int_S \omega=\int_D \omega(p(v))(p^{(1)},\ldots,p^{(k)})\,dv$, where $p^{(i)}$ is the i-th column of $p'(v)$.

Integration over differential forms can be defined without the notion of a tensor, for example as in [Rudin]. A differential form may be defined to be a formal expression $\sum_T \omega_T$, where each ω_T is a scalar field; the integral is defined as before. See [Rudin] for further details.

Theorem 1. Suppose $p_1 : U_1 \mapsto \mathcal{R}_n$ is a regular k-surface, $D_1 \subseteq U_1$ is Jordan measurable, and $p_1' \restriction D_1$ is bounded. Suppose $U_2 \subseteq \mathcal{R}^k$ is open. and $q : U_2 \mapsto U_1$ is a C_1 diffeomorphism. Let $p_2 = p_1 \circ q$ and let $D_2 = q^{-1}[D_1]$. Suppose $q' \restriction D_2$ is bounded. Suppose ω is a k-form with domain W, and $p_1[D] \subseteq W$. For $i = 1, 2$ let J_T^i denote the Jacobian fpr p_i. Then the following hold.

a. If $\det(q'(\nu)) > 0$ for all $\nu \in D_2$ then $\sum_T \int_{D_1}(\omega_T \circ p_1)\cdot J_T^1$ equals $\sum_T \int_{D_2}(\omega_T \circ p_2)\cdot J_T^2$.

b. If $\det(q'(\nu)) < 0$ for all $\nu \in D_2$ then $\sum_T \int_{D_1}(\omega_T \circ p_1)\cdot J_T^1$ equals $\sum_T -\int_{D_2}(\omega_T \circ p_2)\cdot J_T^2$.

Proof: As in the proof of lemma 14.5,

$\sum_T \int_{D_1} \omega_T(p_1(x))\cdot J_T^1(x)\,dx$ equals

$\sum_T \int_{q^{-1}[D_1]} \omega_T(p_1(q(x)))\cdot J_T^1(p_1(q(x)))\,|\det(q'(x)|\,dx$, which equals

$\sum_T \pm\int_{q^{-1}[D_1]} \omega_T(p_1(q(x)))$

$\cdot J_T^1(p_1(q(x)))\det(q'(x))\,dx$,

with the sign as in the statement of the theorem. Using the chain rule for each row of $[p_{t_i}^{\partial_j}(v)]$ it follows that $J_T^1(p_1(q(x)))\det(q'(x))$ equals $J_T^2(p_2(x))$. The theorem follows. QED.

Theorem 2. Suppose $q : U_2 \mapsto U_1$ is a C_1 diffeomorphism between open subsets of $\mathcal{R}^k$. Suppose $D_2 \subseteq U_2$ is connected. Then either $\det(q'(\nu)) > 0$ for all $\nu \in D_2$, or $\det(q'(\nu)) < 0$ for all $\nu \in D_2$.

Proof: The function $v \mapsto \det(q'(v))$ is continuous, being a multivariable polynomial composed with some projection maps. By theorem 5.15 $\det(q'[D_2])$ is a connected subset of $\mathcal{R}$, whence by theorem 5.16 it is an interval. By lemma 13.5 it cannot contain 0. QED.

For theorem 5 below, an additional fact about partial derivatives is required, which has many other uses in vector calculus. First, some additional facts concerning multilinear maps will be given.

Suppose X, Y are vector spaces over a field F and $\phi \in L(X; L(X;\cdots L(X;Y)\cdots))$. Let $\bar\phi(x_1,\ldots,x_n)=\phi(x_1)(x_2),\cdots(x_r)$; then it is readily seen that $\bar\phi \in L(X^r;Y)$, and the map $\phi \mapsto \bar\phi$ is an isomorphism of vector spaces; indeed $\phi(x) = \bar{\phi}_x$ where $\phi_x(y) = \phi(x,y)$, etc. In particular, if f is a differentiable function, $f^{(r)}$ may be considered as an r-ary multilinear function. An r-ary multilinear function is said to be symmetric if $f(x_{\pi(i_1)}, \cdots, x_{\pi(i_r)}) = f(x_{i_1}, \cdots, x_{i_r})$ for all $\pi \in \mathrm{Sym}_r$.

To simplify the notation, for the rest of the chapter the strong norm of an element of $\phi \in L(\mathcal{R}^m; \mathcal{R}^n)$ will be denoted $|\phi|$. Recall that by definition, $|\phi| = \inf\{c \in \mathcal{R}^{\geq} : |\phi(x)| \leq c|x| \text{ for all } x \in X\}$. It may alternatively be characterized as $\sup\{|\phi(x)|/|x| : |x| \neq 0\}$, $\sup\{|\phi(x)| : |x| = 1\}$, or $\sup\{|\phi(x)| : |x| \leq 1\}$ (exercise 2).

In the case $r = 2$, $|\phi| \leq r$ iff for all x $|\phi(x)| \leq r|x|$ iff for all x $|\phi_x| \leq r|x|$ iff for all x, y $|\phi_x(y)| \leq r|x||y|$ iff for all x, y $|\bar{\phi}(x,y)| \leq r|x||y|$. Thus, defining $|\bar{\phi}| = \inf\{r : |\phi(x,y)| \leq r|x||y| \text{ for all } x, y\}$ yields a norm on $L((\mathcal{R}^m)^2; \mathcal{R}^n)$, which makes the map $\phi \to \bar{\phi}$ an isometry (i.e., distance-preserving). Similarly, defining $|\bar{\phi}| = \inf\{r : |\bar{\phi}(x_1, \ldots, x_n)| \leq r|x_1|\cdots|x_n| \text{ for all } x_1, \ldots, x_n\}$ for $\bar{\phi} \in L((\mathcal{R}^m)^r; \mathcal{R}^n)$ yields a norm under which the vector space isomorphism is an isometry; further, $|\bar{\phi}| = \sup\{|\bar{\phi}(x_1, \ldots, x_n)| : |x_i| \leq 1 \text{ for all } x_1, \ldots, x_n\}$ (exercise 3).

Lemma 3. Suppose $U \subseteq \mathcal{R}^m$ is an open subset, and $f : U \mapsto \mathcal{R}^n$ is twice differentiable at the point $x_0 \in U$. Then $f''(x_0)$, considered as a bilinear form, is symmetric.

Proof: Suppose r is such that$B^{\mathrm{cl}}_{x_0,r} \subseteq U$, and $s, t \in \mathcal{R}^m$ are such that $|s|, |t| \leq r/2$. Let $A = f'(x_0 + \xi s + t) - f(x_0) - f''(x_0)(\xi s + t)$, $B = f'(x_0 + \xi s) - f(x_0) - f''(x_0)(\xi s)$, and $C = f'(x_0 + t) - f(x_0) - f''(x_0)$. By the assumption that $f''(x_0)$ exists, given $\epsilon > 0$ there exists an $r' \leq r$ such that for s, t with $|s|, |t| \leq r'$ and $0 \leq \xi \leq 1$, $|A| \leq \epsilon(|s| + |t|)$, $|B| \leq \epsilon(|s|)$, and $|C| \leq \epsilon(|t|)$.

Let g be the function where $g(\xi) = f(x_0 + \xi s + t) - f(x_0 + \xi s)$; then $g'(\xi) - f''(x_0)(t)(s) = (A - B)(s)$ and $g'(0) - f''(x_0)(t)(s) = Cs$. Then $|g'(\xi) - g'(0)| = |(g'(\xi) - f''(x_0)(t)(s)) - (g'(0) - f''(x_0)(t)(s))| \leq 2\epsilon|s|(|s| + |t|)$. Applying temma 4.10 to the function $g(\xi) - \xi g'(0)$, $|g(1) - g(0) - g'(0)| \leq 2\epsilon|s|(|s| + |t|)$. Then $|g(1) - g(0) - f''(x_0)(t)(s)| = |(g(1) - g(0) - g'(0)) + (g'(0) - f''(x_0)(t)(s))| \leq 3\epsilon|s|(|s| + |t|)$. Now, $g(1) - g(0) = f(x_0 + s + t) - f(x_0 + s) - f(x_0 + t) + f(x_0)$ is symmetric in s and t, so exchanging s and t, $|g(1) - g(0) - f''(x_0)(s)(t)| \leq 3\epsilon|t|(|s| + |t|)$. Letting $d(s,t)$ denote the bilinear form $|f''(x_0)(t)(s) - f''(x_0)(s)(t)$, it follows that $|d(s,t)| \leq 6\epsilon(|s| + |t|)^2$. This holds for $|s|, |t| \leq r'/2$; multiplying s, t by a scalar and using bilinearity it holds for all s, t.

It follows that $|d| \leq 24\epsilon$, whence $|d| = 0$, whence $d = 0$. QED.

For the following corollary, suppose $U \subseteq \mathcal{R}^m$ is an open subset, and $f : U \mapsto \mathcal{R}^n$ is C_1. In contrast to theorem 4.11, $f_i^{\partial x_j}$ will be used to denote the function from U to $L(\mathcal{R}; \mathcal{R})$. As noted in chapter 5, f^{∂_j} may be used to denote the function $\langle \partial f_1/\partial x_j, \ldots, \partial f_m/\partial x_j \rangle$ from U to

$L(\mathcal{R};\mathcal{R}^m)$. If f is twice differentiable at x_0 then for $1 \leq j,k \leq m$, $\partial f^{\partial j}/\partial x_k$ exists at x_0; this quantity is denoted

$$\frac{\partial^2 f}{\partial x_j \partial x_k}.$$

The above notation is introduced for use in proving theorems. For use in routine calculus, note that

$$\frac{\partial^2 f_i}{\partial x_j \partial x_k}$$

may be considered either to be an element $\langle \xi, \eta \rangle \mapsto c\xi\eta$ of $L(\mathcal{R}^2;\mathcal{R})$, or the scalar c. Varying with $x \in U$, it is computed as a function from $\mathcal{R}$ to $\mathcal{R}$ in the usual way, namely, first differentiate f_i with respect to x_j, then differentiate the result with respect to x_k.

Corollary 4. Suppose f is twice differentiable at the point $x_0 \in U$. Then for $1 \leq j,k \leq m$,

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j}.$$

Proof: By theorem 4.11, it follows that

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = f''(x_0)(e_j)(e_k).$$

The corollary follows from the lemma. QED.

By an inductive argument, lemma 3 may be seen to hold for $f^{(r)}$ for any r; see theorem 8.12.4 of [Dieudonne].

Suppose $\omega = \sum_T \omega_T(x) dx_T$ is a C_1 rank $k-1$ differential form where $1 \leq k \leq n$. Then $d\omega$ is the k-form defined as follows:

$d\omega = \sum_T \sum_j (\partial \omega_T / \partial x_j) dx_j \wedge dx_T$.

Recall that $dx_j \wedge dx_T$ equals 0 if $j \in T$; and that j can be put in its position in the numerical order by changing the sign as many times as j must be transposed with it's right neighbor.

Let Forms$_{kr}$ denote the elements of Forms$_k$ which are C_r; using theorem 4.6, Forms$_{kr}$ may be seen to be a vector subspace of Forms$_k$. Since the derivative of a C_r function is C_{r-1}, for $r \geq 1$ d is a map from Forms$_{k-1,r}$ to Forms$_{k,r-1}$.

Theorem 5. The map d has the following algebraic properties, where formulas are written with d having higher precedence that $\wedge$.

a. It is a linear transformation, in particular $d\omega = \sum_T d(\omega_T dx_T)$.

b. For $\omega_1 \in \mathrm{Forms}_k$ and $\omega_2 \in \mathrm{Forms}_l$, $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$.

c. For $\omega \in \mathrm{Forms}_{k2}$, $dd\omega = 0$

Proof: Part a may be readily verified by direct computation from the definition. For part b,

$$
\begin{aligned}
&d(\omega_{T_1} dx_{T_1} \wedge \omega_{T_2} dx_{T_2}) = \\
&d(\omega_{T_1} \omega_{T_2} dx_{T_1} \wedge dx_{T_2}) = \\
&\textstyle\sum_t (\partial(\omega_{T_1}\omega_{T_2})/\partial x_t) dx_t \wedge dx_{T_1} \wedge dx_{T_2}) = \\
&\textstyle\sum_t (\partial \omega_{T_1}/\partial x_t)\omega_{T_2} dx_t \wedge dx_{T_1} \wedge dx_{T_2} + \\
&\qquad \textstyle\sum_t \omega_{T_1}(\partial \omega_{T_2}/\partial x_t) dx_t \wedge dx_{T_1} \wedge dx_{T_2} = \\
&\big(\textstyle\sum_t (\partial \omega_{T_1}/\partial x_t) dx_t \wedge dx_{T_1}\big) \wedge \omega_{T_2} dx_{T_2} + \\
&\qquad \omega_{T_1} \wedge (-1)^k \big(\textstyle\sum_t (\partial \omega_{T_2}/\partial x_t) dx_t \wedge dx_{T_2} = \\
&d(\omega_{T_1} dx_{T_1}) \wedge \omega_{T_2} dx_{T_2} + (-1)^k \omega_{T_1} dx_{T_1} \wedge d(\omega_{T_2} dx_{T_2})\big).
\end{aligned}
$$

The claim follows by linearity. For part c,

$$
\begin{aligned}
&d(d(\omega_T dx_T)) = \\
&d\big(\textstyle\sum_t (\partial \omega_T/\partial x_t) dx_t \wedge dx_T\big) = \\
&\textstyle\sum_{t,s} (\partial^2 \omega_T/(\partial x_s \partial x_t)) dx_s \wedge dx_t \wedge dx_T,
\end{aligned}
$$

which equals 0 by corollary 4. The claim follows by linearity. QED.

For a helpful example, commonly encountered operations on vector fields when $n = 3$ can be viewed as examples of exterior differentiation.

$k = 1$. Suppose ω is the scalar field $\omega_\emptyset$. Then $d\omega$ is the 1-form $(\partial\omega_\emptyset/\partial x_1)dx_1 + (\partial\omega_\emptyset/\partial x_2)dx_2 + (\partial\omega_\emptyset/\partial x_3)dx_3$. With basis $\langle dx_1, dx_2, dx_3\rangle$ for the 1-forms, $d\omega$ can be identified with the vector $\langle \partial\omega_\emptyset/\partial x_1, \partial\omega_\emptyset/\partial x_2, \partial\omega_\emptyset/\partial x_3\rangle$. This vector is known as the gradient of the scalar field.

$k = 2$. Suppose ω is the 1-form $\omega_1 dx_1 + \omega_2 dx_2 + \omega_3 dx_3$, and let v denote the corresponding vector field. Then $d\omega$ is the 2-form $(\partial\omega_2/\partial x_1 - \partial\omega_1/\partial x_2)dx_1 \wedge dx_2 - (\partial\omega_3/\partial x_1 - \partial\omega_1/\partial x_3)dx_3 \wedge dx_1 + (\partial\omega_3/\partial x_2 - \partial\omega_2/\partial x_3)dx_2 \wedge dx_3$. With the basis $\langle dx_2 \wedge dx_3, dx_3 \wedge dx_1, dx_1 \wedge dx_2\rangle$ for the 2-forms, the vector corresponding to $d\omega$ is $\langle \partial\omega_3/\partial x_2 - \partial\omega_2/\partial x_3, -(\partial\omega_3/\partial x_1 - \partial\omega_1/\partial x_3), \partial\omega_2/\partial x_1 - \partial\omega_1/\partial x_2\rangle$; this vector is known as the curl of v.

$k = 3$. Suppose ω is the 2-form $\omega_1 dx_2 \wedge dx_3 + \omega_2 dx_3 \wedge dx_1 + \omega_3 dx_1 \wedge dx_2$, and let v denote the corresponding vector field. Then $d\omega$ is the 3-form $(\partial\omega_1/\partial x_1 + \partial\omega_2/\partial x_2 + \partial\omega_3/\partial x_3)dx_1 \wedge dx_2 \wedge dx_3$. This corresponds to a scalar field, which is known as the divergence of v.

To conclude this chapter, examples of differential forms mentioned above will be given. Recall the inner product operation on defined in chapter 2, which is denoted $x \cdot y$; x and y are vectors (in this discussion they will be elements of $\mathcal{R}^3$), and the result is a scalar. As already seen, the inner product is a symmetric bilinear form. (It is safe to use the same symbol for the inner product of vectors as for multiplication of scalars, since it is clear from context whether the arguments are vectors or scalars.)

The vector product or cross product operation $x \times y$ is defined for vectors $x, y \in \mathcal{R}^3$, by the equation $x \times y = \langle x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1\rangle$. This operation is readily verified to be a bilinear function from $\mathcal{R}^3 \times \mathcal{R}^3$ to $\mathcal{R}^3$; and to be antisymmetric, that is, $y \times x = -x \times y$. (Again, it is safe to use the same symbol for vector product of vectors and Cartesian product of sets.)

The expression $x \cdot (y \times z)$ is so frequently encountered that it is given a name, the "triple product". It is easily seen that the value equals the determinant of the 3×3 matrix whose columns are x, y, and z. Letting t be the linear transformation with this matrix, the triple product is the volume of the parralelipiped $t[C]$ where C is the "unit cube" $\{x : 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3\}$.

A line in $\mathcal{R}^n$ $(n \geq 2)$ may be given as a parameterized curve, by the map $l(t) = x_0 + tv$ where x_0 is a fixed vector called the root point, v is a fixed nonzero vector called the direction vector, and $t \in \mathcal{R}$. Appendix 1 of [DowdBG] presents relevant facts in the case $n = 2$. If v is multiplied by a nonzero scalar, then the set of points $l[\mathcal{R}]$ does not change; but only the rate at which the points are traversed as t varies. A vector is called a unit vector if $|v| = 1$. If v is a nonzero vector then $v/|v|$ is a unit vector, in the "same direction" as v; replacing v by $v/|v|$ does not change the set of points of a line.

Nonzero vectors $x, y \in \mathcal{R}^n$ are said to be normal if $x \cdot y = 0$; again see appendix 1 of [DowdBG] for relevant facts in the case $n = 2$.

A plane in $\mathcal{R}^n$ $(n \geq 3)$ may be given as a parameterized surface, by the map $p(t) = x_0 + t_1v_1 + t_2v_2$ for nonzero v_1, v_2. Suppose for the rest of the paragraph that $n = 3$. A nonzero vector w is said to be normal to the plane if $w \cdot v_1 = 0$ and $w \cdot v_2 = 0$. If this is the case then $w \cdot (x - x_0) = 0$ for every point $x - x_0$ in the plane, and w is said to be normal to the plane. By basic vector algebra, $v_1 \times v_2$ is normal to the plane. Further, letting p be the 3×2 matrix whose columns are v_1 and v_2, direct computation shows that $(v_1 \times v_2) \cdot (v_1 \times v_2) = \det(p^tp)$. From this it follows that $|v_1 \times v_2|$ is the area of the parallelogram $p[S]$ where p indifferently denotes the linear transformation and S is the "unit square" $\{x : 0 \leq x_i \leq 1 \text{ for } i = 1, 2\}$.

Suppose $c : (a, b) \mapsto \mathcal{R}^n$ is a regular curve. Suppose $t_0 \in (a, b)$, and let $x_0 = c(t_0)$ and $C = c[(a, b)]$. By the definition of the derivative, for $t \in (a, b)$ $c(t) \approx c(t_0) + (t - t_0)c'(t_0)$. The right side is a parametrization of a line, called the line tangent to the curve C at the point x_0. It is the unique line passing through x_0 whose parametrization approximates c at t_0, in the well defined sense of approximation given by the definition of the derivative.

Suppose $s : U \mapsto \mathcal{R}^3$ is a regular surface. Suppose $t_0 \in U$, and let $x_0 = s(t_0)$ and $S = s[U]$. As for curves, for $t \in U$ $s(t) \approx c(t_0) + s'(t_0)(t - t_0)$. The right side is a parametrization of a plane, called the plane tangent to the surface S at the point x_0. It is the unique plane containing x_0 which best approximates S at x_0. The "basis vectors" of the plane are the columns of s', namely, $\langle \partial c_1/\partial t_j, \partial c_2/\partial t_j, \partial c_3/\partial t_j \rangle$ for $j = 1, 2$. Below, these vectors will be denoted as $s^{(1)}$ and $s^{(2)}$.

Suppose $s : U \mapsto \mathcal{R}^3$ is a regular surface, $c : (a, b) \mapsto U$ is a regular curve, $S = s[U]$, and $C = (s \circ c)[(a, b)]$. Then C is a regular curve and $C \subseteq S$. As would be expected, the tangent line to C at a point $x \in C$ lies in the tangent plane to S at x (exercise A1).

Suppose v is a vector field with domain $W \subseteq \mathcal{R}^3$, c is a regular curve with domain (a, b), $C = c[(a, b)]$, and $C \subseteq W$. The "line integral" of v along c, or the integral of the tangential component of v, is denoted as $\int_C v \cdot t$, and defined as $\int_a^b (v(c(v)) \cdot t(v))|J^v(v)|\,dv$, where $t(v) = c'(v)/|c'(v)|$. Now, $J^v(v) = c'(v)$, so the integrand equals $v(p(v)) \cdot c'(v)$. Expanding the dot product, the integral may be seen to equal $\int_C \omega$ where ω equals $v_1 dx_1 + v_2 dx_2 + v_3 dx_3$.

Suppose v is a vector field with domain $W \subseteq \mathcal{R}^3$, s is a regular surface with domain U, $D \subseteq W$ is Jordan measurable, $S = s[D]$, and $S \subseteq W$. The "surface integral" of v over S, or the integral of the normal component of v, is denoted as $\int_S v \cdot n$, and defined as $\int_D (v(s(v)) \cdot n(v))|J^v(v)|\,dv$, where $n(v) = s^{(1)}(v) \times s^{(2)}(v)/|s^{(1)}(v) \times s^{(2)}(v)|$. Now, by remarks above, $J^v(v) = s^{(1)}(v) \times s^{(2)}(v)$, so the integrand equals $v(s(v)) \cdot (s^{(1)}(v) \times s^{(2)}(v))$. Expanding the triple product, the integral may be seen to equal $\int_C \omega$ where ω equals $v_1 dx_2 \wedge dx_3 - v_2 dx_1 \wedge dx_3 + v_3 dx_1 \wedge dx_2$, which may also be written $v_1 dx_2 \wedge dx_3 + v_2 dx_3 \wedge dx_1 + v_3 dx_1 \wedge dx_2$.

Exercises.

1. Show the following, for a vector space V over a field F.

a. Let $\chi_{i_1 \cdots i_k}$ be the multilinear map which is 1 at $\langle e_{i_1}, \ldots, e_{i_k} \rangle$, and 0 at any other k-tuple of basis elements; the collection of these comprise a basis for $L(V^k; F)$.

b. $L(V^{k+l}; F)$ is a tensor product of $L(V^k; F)$ and $L(V^l; F)$.

c. Writing V^* for $L(V; F)$, $L(V^k; F)$ equals $V^* \otimes \cdots \otimes V^*$.

d. The basis element $\chi_{i_1} \otimes \cdots \otimes \chi_{i_k}$ corresponds to the multilinear map $\chi_{i_1 \ldots i_k}$

Hint: For part b, let $t(f_1, f_2)$ be the function such that

$$f(e_1, \ldots, e_k, e_{k+1}, \ldots, e_{k+l}) = f(e_1, \ldots, e_k) f_2(e_{k+1}, \ldots, e_{k+l}).$$

Then f is multilinear. Let $t(f_1, f_2) = f$; then t is bilinear. Suppose $t' : L(V^k; F) \times L(V^l; F) \mapsto W$ is bilinear. Mapping $\chi_{i_1 \cdots i_k, i_{k+1} \cdots i_{k+l}}$ to

$t'(\chi_{i_1\cdots i_k}, \chi_{i_{k+1}\cdots i_{k+l}})$ induces a unique linear map $l : L(V^{k+l}; F) \mapsto W$.

2. Prove that the strong norm may be alternatively characterized as indicated in the chapter. Hint: Let B_1 be the value according to the original definition, and let $B_2 - B_4$ be the values according to the alternative definitions in order. Then $B_1 \leq r$ iff $\forall x(|f(x)| \leq r|x|)$, so $B_1 > r$ iff $\exists x(|f(x)| \leq r|x|)$ iff $\exists x \neq 0(|f(x)|/|x| > r)$, iff $B_2 > r$. Equivalence of definitions 3 and 4 is straightforward.

3. Provide further details regarding the properties of the operator norm on $L((\mathcal{R}^m)^r; \mathcal{R}^n)$.

A1. Prove that the tangent to a curve lying in a surface at a point on the curve lies in the tangent plane to the surface at the point. Hint: The tangent plane at x equals $s'(x)[\mathcal{R}^2]$, and $(s \circ c)'(t) = s'(c(t))(c'(t))$.

17. The pullback operation.

The pullback operation is a technical device which will be useful in the next chapter. A technical device from matrix algebra is useful in the discussion.

Let $[n]$ denote the set $\{1, \ldots, n\}$. For $0 \leq k \leq n$ let $\binom{[n]}{k}$ denote the set of k-element subsets of $\{1, \ldots, n\}$; its size is $\binom{n}{k}$. Although not necessary, when using this set as an index set it may be assumed to be ordered in some order; often the lexicographic order is used.

Suppose M is an $n \times m$ matrix, $0 \leq k \leq n, m$, $R \in \binom{[n]}{k}$, $R = \{i_1 < \cdots < i_k\}$, $S \in \binom{[m]}{k}$, $S = \{j_1 < \cdots < j_k\}$. Let M^{RS} be the matrix where $M^{RS}_{rs} = M_{i_r j_s}$. Let $M^{(k)}$ denote the $\binom{n}{k}$ by $\binom{m}{k}$ matrix, where $M^{(k)}_{RS} = \det(M^{RS})$ (if $k = 0$ M^{RS} is the "empty matrix" and $M^{(0)}$ is taken to be the scalar 1). Usage varies, but a matrix of the form M^{RS} is called a minor of M; and $M^{(k)}$ the kth compound of M. The following lemma, or its corollary, is called the Binet-Cauchy theorem.

Lemma 1. Suppose A is a $k \times n$ matrix and B is an $n \times k$ matrix, where $k \leq n$. Write K for $[k]$ and B for $[n]$. Then $\det(AB) = \sum_S \det(A^{KS}) \det(B^{SK})$, where S ranges over $\binom{N}{k}$.

Proof: In the following let μ denote an injection.

$$\begin{aligned}\det(AB) &= \sum_\sigma \mathrm{sg}(\sigma) \prod_{1\le l\le k} (AB)_{l,\sigma(l)} \\ &= \sum_\sigma \mathrm{sg}(\sigma) \prod_{1\le l\le k} \sum_{1\le r\le n} A_{lr}B_{r,\sigma(l)} \\ &= \sum_\sigma \mathrm{sg}(\sigma) \sum_{\mu:K\mapsto N} \prod_{1\le l\le k} A_{l,\mu(l)}B_{\mu(l),\sigma(l)} \\ &= \sum_S \sum_\sigma \mathrm{sg}(\sigma) \sum_{\mu:K\mapsto N,\mu[K]=S} \prod_{1\le l\le k} A_{l,\mu(l)}B_{\mu(l),\sigma(l)} \\ &= \sum_S \sum_\sigma \mathrm{sg}(\sigma) \sum_\pi \prod_{1\le l\le k} A^{KS}_{l,\pi(l)}B^{SK}_{\pi(l),\sigma(l)}.\end{aligned}$$

But it is easily seen that the sum over σ equals $\det(A^{KS})\det(B^{SK})$. QED.

Corollary 2. Suppose A is an $l \times m$ matrix, B is an $m \times n$ matrix, and $k \le l, m, n$. Then $(AB)^{(k)} = A^{(k)}B^{(k)}$.

Proof: $(AB)^{(k)}_{RT} = \det((AB)_{RT})$, and, letting S range over $\binom{[m]}{k}$, $(A^{(k)}B^{(k)}))_{RT} = \sum_S A^{(k)}_{RS}B^{(k)}_{ST} = \sum_S \det(A_{RS})\det(B_{ST})$.
These are equal by lemma 1. QED.

Suppose $V \subseteq \mathcal{R}^m$ and $W \subseteq \mathcal{R}^n$ are open subsets, and $t : V \mapsto W$ is a C_1 function. Let Forms_k^W denote the k-forms on W, and Forms_k^V the k-forms on V. Then $t^* : \mathrm{Forms}_k^W \mapsto \mathrm{Forms}_k^V$ is defined as follows. If $\omega = \sum_R \omega_R dx_R$ where R ranges over $\binom{[n]}{k}$, then $t^*(\omega) = \sum_S \tilde{\omega}_S dx_S$ where S ranges over $\binom{[m]}{k}$, and $\tilde{\omega}_S = \sum_R (\omega \circ t) \cdot t'^{(k)}_{RS}$.

For the following, it is useful to let dx_i denote ambiguously the basis element, or the 1-form $1 \cdot dx_i$. Note that $f \wedge dx_S = f dx_S$.

Theorem 3. t^* is the unique function from $\mathrm{Forms}_k^W \mapsto \mathrm{Forms}_k^V$ such that the following hold.

a. $t_*(\omega_\emptyset) = \omega_\emptyset \circ t$.
b. $t^*(dx_i) = \sum_{j=1}^m (\partial t_i/\partial x_j)dx_j$.
c. t^* is linear.
d. $t^*(\omega_1 \wedge \omega_2) = t^*(\omega_1) \wedge t^*(\omega_2)$.

Proof: Parts a-c follow by direct computation from the definition. For part d, it is readily verified that dt_i equals $t^*(x_i)$. Let R range over $\binom{[n]}{k}$ and S over $\binom{[m]}{k}$. For $R = \{i_1 < \cdots < i_k\}$ let $dt_R = dt_{i_1} \wedge \cdots \wedge dt_{i_k}$. One verifies that $dt_R = \sum_S t'^{(k)}_{RS} dx_S$, and $t^*(\omega) = \sum_R (\omega_R \circ t) dt_R$. Similarly, for $\lambda \in \mathrm{Forms}_l^W$, letting R' range over $\binom{[n]}{l}$, $t^*(\lambda) = \sum_{R'} (\lambda_{R'} \circ t)dt_{R'}$. Then $t^*(\omega) \wedge t^*(\lambda) = \sum_R \sum_{R'} (\omega_R \circ t)(\lambda_{R'} \circ t) dt_R \wedge dt_{R'}$. But

$t^*(\omega \wedge \lambda) = \sum_R \sum_{R'} (\omega_R \lambda_{R'}) \circ t \, dt_R \wedge dt_{R'}$, and part d follows. Finally, a value for $t^*(\omega)$ for any ω is clearly determined by the requirements of parts a-d. QED.

Theorem 4.

a. If t is C_2 then $d(t^*(\omega)) = t^*(d\omega)$.

b. $(t_1 \circ t_2)^* = t_2^* \circ t_1^*$.

Suppose p is a C_1 k-surface with domain U, $D \subseteq U$ is Jordan measurable, and $p[U] \subseteq V$. Let $E = p[D]$.

c. $\int_{t[E]} \omega = \int_E t^*(\omega)$.

Proof: Part a may be proved by induction on k. For $k = 0$,
$d(t^*(\omega_\emptyset)) = d(\omega_\emptyset \circ t) = \sum_{j=1}^m (\partial(\omega_\emptyset \circ t)/\partial x_j) dx_j$
$= \sum_{j=1}^m \sum_{i=1}^n ((\partial \omega_\emptyset/\partial x_j) \circ t)(\partial t_i/\partial x_j) dx_j$
$= t^*(\sum_{i=1}^n (\partial \omega_\emptyset/\partial x_i) dx_i = t^*(d\omega_\emptyset)$.
For $0 < k < n$, it suffices to consider forms $\lambda = d_{x_i} \wedge \omega$ where $\omega \in \text{Forms}_k^W$. For such, using theorem 16.5, $d\lambda = ddx_i \wedge \omega - dx_i \wedge d\omega = -dx_i \wedge d\omega$, and hence $t^*(d\lambda) = -t^*(dx_i) \wedge t^*(d\omega)$. Using the induction hypothes, $t^*(d\lambda) = -t^*(dx_i) \wedge d(t^*(\omega))$. Using theorem 16.5 and an observation above, $d(t^*(dx_i)) = ddt_i = 0$. Thus, $t^*(d\lambda) = d(t^*(dx_i)) \wedge t^*(\omega) - t^*(dx_i) \wedge d(t^*(\omega)) = d(t^*(dx_i) \wedge t^*(\omega)) = d(t^*(dx_i \wedge \omega)) = d(t^*(\lambda))$. Part b follows by a straightforward computation using corollary 2 (it is helpful to think of ω as a row vector of length $\binom{n}{k}$; $t^*(\omega)$ then equals $(\omega \circ t)t'^{(k)})$. For part c, note that $J_R^p = t'^{(k)}_{RK}$, and proceed as for part b. QED.

18. Stokes' theorem for cell chains.

Stokes' theorem states that, for certain k-surfaces V which have a $(k-1)$-surface S as an "oriented" boundary, the integral of a $(k-1)$-form ω over S equals the integral of $d\omega$ over V. There are various versions, depending on the conditions placed on the surfaces. In a general version, V is a "compact orientable manifold with boundary" S (theorem 5.5 of [Spivak]).

Here the theorem will be proved for k-cells. It is convenient to prove it more generally, for "cell chains". Chains are a technical device which have numerous uses in topology.

The terminology used for cell chains varies widely, and that used here will be the author's own. For ease of notation, let I^k denote $\{x \in \mathcal{R}^k : 0 \le x_i \le 1 \text{ for } 1 \le i \le k\}$. In this chapter, unless otherwise specifie, W denotes an open subset of $\mathcal{R}^n$, and ω a k-form with domain W.

A C_1 k-cell in W is defined to be a C_1 surface $c : U \mapsto W$, where $I^k \subseteq U$. Cells are of interest, in that integrals may be defined over both $c[I^k]$ and its boundary; the definition is given by "pulling back" to I^k.

The pullback requires c to be at least differentiable, although some facts of interest hold for C_0 cells. A C_1 k-cell chain in W is a formal integer linear combinations of C_1 k-cells in W, that is, an element of the free $\mathcal{Z}$-module generated by such cells. Chains are of interest, in that the boundary of a k-cell can be given as a $(k-1)$-chain.

For $1 \le d \le n$ and $s \in \{0,1\}$ let $\mu_{ds} : \mathcal{R}^{k-1} \mapsto \mathcal{R}^k$ be the function mapping x to $\langle x_1, \ldots, x_{d-1}, s, x_d, \ldots, x_{n-1}\rangle$. Given a C_1 k-cell $c : U \mapsto W$, let $c_{ds} = c \circ (\mu_{ds} \restriction \mu_{ds}^{-1}[U])$. I^{k-1} is a subset of $\mathrm{Dom}(c_{ds})$, whence c_{ds} is a C_1 $(k-1)$-cell; these $2k$ cells are called the "boundary faces" of c. The boundary ∂c of a cell c is defined to be $\sum_{d,s}(-1)^{(d+s)} c_{ds}$. For a chain $C = \sum_c n_c c$, the boundary is defined by linearity, namely, $\partial C = \sum_c n_c \partial c$.

Theorem 1. The map ∂ from k-chains in W to $(k-1)$-chains in W has the following algebraic properties.

a. It is a linear transformation.

b. For any k chain C where $k \ge 2$, $\partial\partial C = 0$.

Proof: Part a is readily verified from the definition. For part b, for a cell c, $\partial\partial c = \sum_{e,t}\sum_{d,s}(-1)^{e+t+s+s}(c_{ds})_{et}$. Letting ν_{ds} denote μ_{ds} when the domain is $\mathcal{R}^{k-2}$, it is readily verified that for $1 \le d < e \le n$, $\mu_{et} \circ \nu_{ds} = \mu_{ds} \circ \nu_{e-1,t}$. It follows that $(c_{et})_{ds} = (c_{ds})_{e-1,t}$, whence that $\partial\partial c = 0$. The claim for chains follows by linearity. QED.

As noted earlier in the chapter, the definition of the integral of a k-form ω over a C_1 k-cell c is essentially that already given for a k-surface, namely, $\int_c \omega = \sum_T \int_{I^k} (\omega_T \circ c) \cdot J_T^c$. For a chain $C = \sum_c n_c c$, the integral is defined by linearity, namely, $\int_C \omega = \sum_c n_c \int_c \omega$.

Lemma 2. Suppose ω is a $(k-1)$-form on an open set U containing I^k. Let ι denote the identity function on U. Then $\int_\iota d\omega = \int_{\partial i} \omega$.

Proof: It suffices to consider ω of the form $\omega_S dx_S$ where $S = K - \{i\}$ for $K = [k]$ and $i \in K$. Then $d\omega = (-1)^{i-1}\frac{\partial \omega_S}{\partial x_i} dx_K$. Let $\mathcal{I}_{ds}$ denote $\int_{I^{k-1}} \omega_S(x_1, \ldots, x_{d-1}, s, x_{d+1}, \ldots, x_k)$. Then, using theorems 8.12 and 8.13, $\int_\iota d\omega = (-1)^{i-1}\int_{I^k} \frac{\partial \omega_S}{\partial x_i} = (-1)^{i-1}\int_{I^{k-1}} \omega_S(x_1, \ldots, 1, \ldots, x_k) - \omega_S(x_0, \ldots, 1, \ldots, x_k) = (-1)^{i-1}(\mathcal{I}_{i1} - \mathcal{I}_{i0})$. But $\int_{\partial\iota} \omega = \sum_{ds}(-1)^{(d+s)}$ $\int_{\iota_{ds}} \omega_S dx_S = \sum_{ds}(-1)^{(d+s)}\int_{I^{k-1}}(\omega_S \circ \mu_{ds}) \cdot J_S^{\mu_{ds}}$. It is readily verified that $J_S^{\mu_{ds}} = \delta_{id}$, whence $\int_{\partial\iota} \omega = (-1)^{i+1}\mathcal{I}_{i1} + (-1)^i \mathcal{I}_{i0}$. QED.

Theorem 3. Suppose C is a C_2 k-cell chain in W and ω is a C_1 $(k-1)$-form on W. Then $\int_C d\omega = \int_{\partial C} \omega$.

Proof: By linearity it suffices to consider the case that C is a single cell c. Writing c as $c \circ \iota$, it follows by theorem 17.4 that $\int_c d\omega = \int_\iota c^*(d\omega) = \int_\iota d(c^*(\omega))$, and $\int_{\partial c} \omega = \int_{\partial i} c^*(\omega)$. The theorem follows by lemma 2 applied to $c^*(\omega)$. QED.

Theorem 3 is of interest because it is an intuitively clear and easily proved version of Stokes' theorem. It is also of interest because it is used as a lemma in proofs of more general versions, for example theorem 5.5 of [Spivak], which should be within reach of students who have read this book. Intermediate versions can be proved, but in view of the preceding remark these are not of great interest. Some remarks on this topic will be given in the additional material.

To conclude this chapter, theorem 3 will be used to prove versions of three "classical" theorems of vector calculus, Green's theorem ($n = 2$, $k = 2$), Stokes' theorem ($n = 3$, $k = 2$), and Gauss', or the divergence, theorem ($n = 3$, $k = 3$). Theorem 3 is sometimes called the general Stokes theorem, to distinguish it from the $n = 3$, $k = 2$ version.

For Green's theorem, $\omega = \omega_1 dx_1 + \omega_2 dx_2$, which corresponds to the vector field $\langle \omega_1, \omega_2 \rangle$; and $d\omega = (\frac{\partial \omega_2}{x_1} - \frac{\partial \omega_1}{x_2}) dx_1 \wedge dx_2$. Suppose c is a C_2 diffeomorphism and $\det(c'(\nu)) > 0$ for $\nu \in I^2$. Using theorem 16.1, $\int_c d\omega = \int_{c[I^2]} \frac{\partial \omega_2}{x_1} - \frac{\partial \omega_1}{x_2}$. Using remarks in chapter 16, $\int_{\partial c} \omega$ is the line integral around $\int_{\partial c}$ in the counterclockwise direction, of the vector field ω.

For Stokes' theorem, $\omega = \omega_1 dx_1 + \omega_2 dx_2 + \omega_3 dx_3$, which corresponds to the vector field $\langle \omega_1, \omega_2, \omega_3 \rangle$; and $d\omega = (\partial \omega_2 / \partial x_1 - \partial \omega_1 / \partial x_2) dx_1 \wedge dx_2 - (\partial \omega_3 / \partial x_1 - \partial \omega_1 / \partial x_3) dx_3 \wedge dx_1 + (\partial \omega_3 / \partial x_2 - \partial \omega_2 / \partial x_3) dx_2 \wedge dx_3$. As noted in chapter 16, the vector field $d\omega$ is the curl of the vector field ω. Suppose c is a regular C_2 parametrization. Then $\int_c d\omega$ is the surface integral over $c[I^2]$ of the normal component of the curl, and $\int_c d\omega$ is the line integral around $\int_{\partial c}$ of ω. Being "inherited" from I^2, the orientation of ∂c around the normal may be seen to be counterclockwise. A detailed argument, indeed statement, is surprisingly involved; but considering the map $\langle x_1, x_2 \rangle \mapsto \langle x_1, x_2, 0 \rangle$, note that $\langle 1, 0, 0 \rangle \times \langle 0, 1, 0 \rangle = \langle 0, 0, 1 \rangle$.

For Gauss' theorem, $\omega = \omega_{23} dx_2 \wedge dx_3 + \omega_{31} dx_3 \wedge dx_1 + \omega_{12} dx_1 \wedge dx_2$, which corresponds to the vector field $\langle \omega_{23}, \omega_{31}, \omega_{12} \rangle$; and $d\omega = (\frac{\partial \omega_{23}}{\partial x_1} + \frac{\partial \omega_{31}}{\partial x_2} + \frac{\partial \omega_{12}}{\partial x_3}) dx_1 \wedge dx_2 \wedge dx_3$. As noted in chapter 16, the scalar $d\omega$ is the divergence of the vector field ω. Suppose c is a C_2 diffeomorphism. By an argument as for Green's theorem, $\int_c d\omega$ is the "volume integral" of the divergence of ω over $c[I^3]$. $\int_c d\omega$ is the sum of the surface integrals over the faces of c of the normal component of ω. The normal to each face may be seen to be the "outer" normal, that pointing "away" from $c[I^3]$; this may be readily verified for $\partial \iota$ (exercise 1).

Additional material.

An argument will be outlined that Gauss' theorem should hold for the closed ball $B = \{w \in \mathcal{R}^3 : |w| \leq 1\}$. The topological boundary is the 2-sphere $S = \{w \in \mathcal{R}^3 : |w| = 1\}$ (theorem 5.3). The argument involves a "cellulation" of B, by 7 cells.

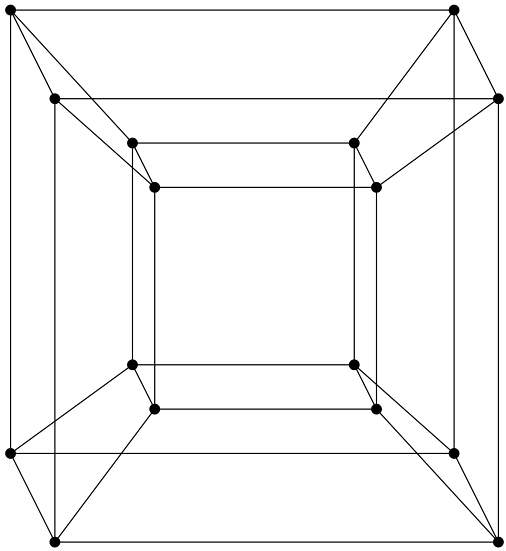


Figure 1

Figure 1 shows how the cells are arranged, with a central cell and 6 surrounding cells. Each of the surrounding cells can be deformed so that it outer surface is a section of S. This can be done so that for each common face, the two copies have opposite orientation, so that these terms in the integral of the boundary chain cancel.

Since a boundary face of I^k has measure 0, the integral over a cell, or boundary face, should equal the integral over its interior, in any reasonable definition of the integral. The sum of the volume integrals of the cells should equal the volume integral over B. The sum of the surface integrals of the outer faces should equal the surface integral over S, since the normals all point outward.

Providing further details is left as exercise A1.

This method clearly applies more generally. Rather than cells, simplexes can be used, where I^k is replaced by the set $S_k = \{u \in \mathcal{R}^k : x_i \geq 0$ for $1 \leq i \leq k$ and $\sum_i x_i \leq 1\}$. It is a fact that any closed C_1 manifold in $\mathcal{R}^n$ can be "triangulated"; see [Whitney] for a proof.

Exercises.

1. Show that the surface normals of $\partial\iota$ point outward. Hint: For example, for the bottom face $d = 3$, $s = 0$, the normal is $\langle 0, 0, 1\rangle$, and the face occurs negatively in the boundary chain.

A1. Provide further details for the argument of the addtional material.

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References to web pages at en.wikipedia.org/wiki/ are of the form [wikiXxx] where Xxx is translated as follows.

AnHol	Analyticity_of_holomorphic_functions
Axiom	Axiom_(computer_algebra_system)
Compact	Compact_space
ConstE	E_(mathematical_constant)
IntSub	Integration_by_substitution
IntLists	Lists_of_integrals
SpaceFil	Space-filling_curve
SymbInt	Symbolic_integration
UnbO	Unbounded_operator
UnivAlg	Universal_algebra
WeiF	Weierstrass_function

Some additional references are as follows.

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